Sequences

Set:

A set is a collection of well-defined objects.
Example: $A = \{odd\ natural\ numbers\ greater\ than\ 0\ and\ less\ than\ 14\}$
$= \{1,3,5,7,9,11,13\}$

Finite set:

A set is said to be finite if the number of elements in it is finite.
Example: $A = \{set\ of\ vowels\}$
$= \{a, e, i, o, u\}$. The number of elements it the set is 5.

Infinite set:

A set is said to be an infinite set if the number of elements in it is infinite.
Example: $A = \{number\ of\ stars\ in\ the\ night\ time\}$

Sequence:

A sequence is a set of numbers which has a 1-1 correspondence with the set of positive integers.

(OR)

Let $f: \mathbb{N} \to \mathbb{R}$ be a function and $f(n) = a_n$. Then $a_1, a_2, a_3 \dots, a_n, \dots$ is called the sequence in $\mathbb{R}$ determined by the function f and is denoted by (a_n) or $\{a_n\}$. $a_n$is called the n^{th} term of the sequence. The range of the function f which is the subset of $\mathbb{R}$ is called the range of the sequence.

Note:

The term of the sequence need not be distinct and the range of the sequence may be finite or infinite.

Examples:

1) The function $f: \mathbb{N} \to \mathbb{R}$ given by $f(n) = n$ determines the sequence $1,2,3 \dots, n, \dots$
2) The function $f: \mathbb{N} \to \mathbb{R}$ given by $f(n) = n^2$ determines the sequence $1,4,9 \dots, n^2, \dots$
3) The function $f: \mathbb{N} \to \mathbb{R}$ given by $f(n) = (-1)^n$ determines the sequence $-1,1,-1,1, \dots$ and the range of the sequence is $\{-1,1\}$.
4) The function $f: \mathbb{N} \to \mathbb{R}$ given by $f(n) = (-1)^{n+1}$ determines the sequence $1,-1,1,-1, \dots$ and the range of the sequence is $\{-1,1\}$.

Note:

However$(-1)^n$ and $(-1)^{n+1}$ are different sequences.

5) The function $f: \mathbb{N} \to \mathbb{R}$ given by $f(n) = 1$ determines the sequence $1,1,1, \dots .1$. The range is $\{1\}$. This type of sequence is called as a constant sequence.
6) The function $f: \mathbb{N} \to \mathbb{R}$ given by $f(n) = 2n + 3$ determines the sequence $5,7,9, ..2n + 3, \dots$
7) The function $f: \mathbb{N} \to \mathbb{R}$ given by $f(n) = \frac{1}{n}$ determines the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots..$
8) The function $f: \mathbb{N} \to \mathbb{R}$ given by $f(n) = \frac{n-1}{2n+3}$ determines the sequence $0, \frac{1}{7}, \frac{2}{9}, \frac{3}{11}, \frac{4}{13}, \dots..$
9) Thefunction $f: \mathbb{N} \to \mathbb{R}$ given by $f(n) = -n$ determines the sequence $-1, -2, -3, \dots, -n, \dots$
10) The function $f: \mathbb{N} \to \mathbb{R}$ given by $f(n) = x^{n-1}$ determines the sequence$1, x, x^2, x^3, \dots, x^n, \dots$

Fibonacci Sequence

A sequence can also be described by specifying the first few terms and stating a rule for determining a_n in terms of the previous terms of the sequence.

Example: $a_1 = 1, a_2 = 1, a_n = a_{n-1+}a_{n-2}$ and therefore the sequence is $1,1,2,3,5,8,13, \ldots$ The sequence is called the Fibonacci sequence.

Limit of the sequence:

Definition:

A sequence (a_n) is said to tend limit l, when given any positive numbers ϵ, however small we can always find an integer N such that $|a_n - l| < \epsilon$ for all $n \geq N$.

Note:

1) Here$l$ is the limit of the sequence and it is expressed as $\lim_{n\to\infty} a_n = l$ or $(a_n) \to l$.
2) $|a_n - l|$ means the numerical value of $a_n - l$.
3) $If\ |a_n - l| < \epsilon$, it can be easily seen that$l - \epsilon < a_n < l + \epsilon$.

Example:

1) The limit of $\left(\frac{1}{n}\right)$ is 0.
2) The limit of $\left(\frac{n-1}{2n+3}\right)$is $\left(\frac{1}{2}\right)$.
3) The limit of $\left(2 + \frac{(-1)^n}{n}\right)$is 2.
4) The limit of $\left(\frac{n+1}{n}\right)$is 1.

Convergence sequence:

A sequence which tends to a finite limit is said to converge and is called a convergent sequence.

Example:

The sequence $\left(\frac{1}{n}\right), \left(\frac{n+1}{n}\right)$ and $\left(\frac{n-1}{2n+3}\right)$ are all convergent sequences.

Divergent Sequence:

A sequence (a_n) is said to diverge to infinity, if given any real number $k > 0$ there exists a $m \in \mathbb{N}$ such that $a_n > k, \forall\ n \geq m$. We can write this as $(a_n) \to \infty$.

(OR)

In the first place, the terms $a_1, a_2, a_3 \ldots, a_n, \ldots$ mmay have the property that if any positive number m, however large it may be, there is a positive integer N so that $a_n > m$ when $n \geq N$.

$$\lim_{n\to\infty} a_n = \infty$$

Example:

The sequence $(n), (2n), (n^2)$ and $(n^3)$are all divergent sequences.

In the second place, the terms may have the property that if any negative number $-m$ is chosen, however large m may be, there is a positive integer N so that $a_n < -m$ when $n \geq N$.

$$\lim_{n\to\infty} a_n = -\infty$$

Oscillating sequence:

When the sequence does not converge and does not diverge to $\pm\infty$, it is said to oscillate.

Example: $(-1)^n, (-1)^{n+1}$

Note:

If x, a, l be the three numbers such that $|x - a| < l$, then $a - x < l < a + l$.

Theorem 1:

A sequence (a_n) cannot converge to two distinct limits.

Proof:

Let the sequence (a_n)converges to two distinct limits l &l_1.

ϵ is a finite quantity.

$|a_n - l| < \epsilon$ for all $n \geq N$ and $|a_n - l_1| < \epsilon$ for all $n \geq N$.

Now $|a_n - l| < \epsilon$ for all $n \geq N \Rightarrow l - \epsilon < a_n < l + \epsilon$ for all $n \geq N$.

Therefore N is a finite quantity depending on ϵ. Hence there are only a finite number of terms of the sequence outside the interval $(l - \epsilon, l + \epsilon)$.

Let ϵ is less than ${}^1/_2\, |l - l_1|$. Then the interval $(l_1 - \epsilon, l_1 + \epsilon)$ lies outside the interval

$(l - \epsilon, l + \epsilon)$. Hence a finite number of terms lies in the interval $(l_1 - \epsilon, l_1 + \epsilon)$. This contradicts that the sequence has also the limit l_1.

Thus the sequence (a_n) cannot converge to two distinct limits.

Theorem 2:

If (a_n) and (b_n) are convergent sequences, then $(a_n \pm b_n)$ is also a convergent sequence.

Proof:

Let (a_n) and (b_n) be two convergent sequences.

Since $(a_n)$is a convergent sequence, let it converge to a, a finite quantity.

$$\lim_{n\to\infty} a_n = a$$

Since $(b_n)$is a convergent sequence, let it converge to b, a finite quantity.

$$\lim_{n\to\infty} b_n = b$$

Let ϵ be an arbitrary positive integer, then there exist numbers N_0 & N_1 depending on ϵ such that

$$|a_n - a| < \epsilon \text{ for all } n \geq N_0 \text{ and } |b_n - b| < \epsilon \text{ for all } n \geq N_1.$$

Let N be greater than N_0 & N_1. Then

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |a_n - a + b_n - b| \\ &\leq |a_n - a| + |b_n - b| \\ &< \epsilon + \epsilon = 2\epsilon, \text{for all } n \geq N \end{aligned}$$

Since ϵ is an arbitrary,$\lim_{n\to\infty}(a_n + b_n) = a + b$

Similarly, $\lim_{n\to\infty}(a_n - b_n) = a - b$ and hence $\lim_{n\to\infty}(a_n \pm b_n) = a \pm b$.

Corollary:

1) If (a_n) is a convergent sequence and (b_n) is a divergent sequence, then$(a_n + b_n)$is a divergent sequence.
2) If (a_n) and (b_n) are both divergent sequence, then$(a_n + b_n)$may be divergent or convergent or oscillate.
3) If (a_n) diverges to ∞ and (b_n) diverges to $-\infty$, then$(a_n + b_n)$may behave in any way.i.e., it may converge to a limit, may oscillate or diverge.

Theorem 3: If $(a_n) \to a$ and $(b_n) \to b$, then $(a_n b_n) \to ab$.

Proof:

Given $(a_n) \to a$ and $(b_n) \to b$.

Let $a_n = a + \lambda_n$ *and* $b_n = b + \mu_n$.

Since $(a_n) \to a$ and $(b_n) \to b$, λ_n and μ_n tend to 0.

$$|\lambda_n| < \epsilon, n \geq N_0 \text{ and } |\mu_n| < \epsilon, n \geq N_1.$$

Let N be greater than $N_0$and N_1.

$$\text{Now} \quad a_n b_n = (a+\lambda_n)(b+\mu_n) = ab + a\mu_n + b\lambda_n + \lambda_n\mu_n\mu_n$$
$$a_n b_n - ab = a\mu_n + b\lambda_n + \lambda_n\mu_n$$
$$|a_n b_n - ab = a\mu_n + b\lambda_n + \lambda_n\mu_n|$$
$$|a_n b_n - ab| \leq |a\mu_n + b\lambda_n + \lambda_n\mu_n|$$
$$|a_n b_n - ab| \leq |a\mu_n| + |b\lambda_n| + |\lambda_n\mu_n| \quad \text{-------(1)}$$
$$|a_n b_n - ab| \leq |a\mu_n| + |b\lambda_n| + |\lambda_n\mu_n|$$
$$|a\mu_n| < |a|\epsilon, |b\lambda_n| < |b|\epsilon, |\lambda_n\mu_n| < \epsilon^2, n \geq N$$

$$(1) \Rightarrow |a_n b_n - ab| < |a|\epsilon + |b|\epsilon + \epsilon^2 = \epsilon(|a| + |b| + \epsilon)$$

Since both $|a|$ and $|b|$ are finite numbers, $|a_n b_n - ab| < A\epsilon$**,** where A is a positive constant.

Since ϵ is an arbitrary,$A\epsilon$ is also arbitrary and hence

$$|a_n b_n - ab| < \epsilon, n \geq N.$$

Thus$(a_n b_n) \to ab$.

Theorem 4: If $(a_n) \to a$ and $(b_n) \to b, b \neq 0$, then $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$.

Proof:

Given $(a_n) \to a$ and $(b_n) \to b, b \neq 0$.

Let $a_n = a + \lambda_n \text{ and } b_n = b + \mu_n$.

Now,

$$\frac{a_n}{b_n} - \frac{a}{b} = \frac{a+\lambda_n}{b+\mu_n} - \frac{a}{b} = \frac{b(a+\lambda_n) - a(b+\mu_n)}{b(b+\mu_n)} = \frac{ab + b\lambda_n - ab - a\mu_n}{b(b+\mu_n)}$$
$$\Rightarrow \frac{a_n}{b_n} - \frac{a}{b} = \frac{b\lambda_n - a\mu_n}{b(b+\mu_n)}$$

For the corresponding arbitrary positive constants ϵ, there exists numbers $N_0 \& N_1$ such that $|\lambda_n| < \epsilon, \forall\, n \geq N_0$ and $|\mu_n| < \epsilon, \forall\, n \geq N_1$.

Let N be greater than $N_0$and N_1.

Now, $|b(b+\mu_n)| = |b||b+\mu_n|$

Since $|b > 0|$and $|\mu_n| < \epsilon$, we assume that $|b + \mu_n| > \frac{1}{2}|b|$

Thus

$$|b+\mu_n| > \frac{1}{2}|b|^2 = l(say).$$

Then

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \leq \frac{|b\lambda_n| + |a\mu_n|}{|b(b+\mu_n)|} < \frac{|b|\epsilon + |a|\epsilon}{l} < \epsilon\frac{(|b| + |a|)}{l} \Rightarrow \left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \epsilon l_1, \forall\, n \geq N,$$

where l_1 is a fixed positive constant.

Hence $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$.

Theorem 5: Cauchy's 1st theorem on limits

If $a_1, a_2, a_3 \dots, a_n, \dots$ denote a sequence of numbers such that $\lim_{n\to\infty} a_n$exists and is equal to l, then $\frac{a_1+a_2+a_{3+}\dots+a_n}{n}$ also exists and is equal to l.

Proof:

Let $S_n = a_1 + a_2 + a_{3+} \dots + a_n$. Since $(a_n) \to l$, we can find N corresponding to the arbitrary constant ϵ, such that

$$l-\epsilon < a_n < l+\epsilon \text{ for all } n \geq N.$$

$$\text{Now, } l-\epsilon < a_{N+1} < l+\epsilon$$

$$l-\epsilon < a_{N+2} < l+\epsilon$$

$$l-\epsilon < a_{N+2} < l+\epsilon$$

$$\cdots\cdots$$

$$\cdots\cdots$$

$$\cdots\cdots$$

$$l-\epsilon < a_n < l+\epsilon$$

Adding we get,

$$(n-N)(l-\epsilon) < a_{N+1} + a_{N+2} + a_{N+3+} \dots + a_n < (n-N)(l+\epsilon)$$

Adding S_n to this inequality, we have

$$S_N + (n-N)(l-\epsilon) < S_n < S_n + (n-N)(l+\epsilon)$$

Divide by n, we get,

$$\frac{S_n}{n} + \frac{(n-N)(l-\epsilon)}{n} < \frac{S_n}{n} < \frac{S_n}{n} + \frac{(n-N)(l+\epsilon)}{n}$$

$$\frac{S_N}{n} - \frac{N}{n}(l-\epsilon) - \epsilon < \frac{S_n}{n} - l < \frac{S_N}{n} - \frac{N}{n}(l+\epsilon) + \epsilon$$

Since N is a fixed number, $S_N, N(l-\epsilon), N(l+\epsilon)$ are fixed, we can find a number A such that $S_N, N(l-\epsilon), N(l+\epsilon)$ are all less than A.

$$\frac{-2A}{n} - \epsilon < \frac{S_n}{n} - l < \frac{2A}{n} + \epsilon$$

Since A is fixed, we can find positive integer $N_1 > N$depending on ϵ such that $\forall\, n \geq N_1, \frac{2A}{n} < \epsilon \;\; \& \;\; -\frac{2A}{n} > -\epsilon.$

Hence the inequality becomes,

$$-2\epsilon < \frac{S_n}{n} - l < 2\epsilon$$

$$\Rightarrow \left|\frac{S_n}{n} - l\right| < 2\epsilon, \forall\, n \geq N_1.$$

Since ϵ is an arbitrary,2ϵ is also arbitrary and hence

$$\lim_{n\to\infty} \frac{S_n}{n} = l.$$

Dedikind's theorem:

If the system of real numbers is divided into two classes A and B in such a way that

(i) Each classes contains atleast one number.

(ii) Every number belongs to one class or the other.

(iii) Every number in the lower class A is less than every number in class B, then there is number α such that every number less than α belongs to the lower class A and every number greater than αbelongs to the upper class B. α itself will belong to one and only to one of the classes.

Bounded sequences:

Let $a_1, a_2, a_3 \dots, a_n, \dots$ denote a sequence of numbers. Then the sequence is said to be bounded above if there exists a finite number A with the property that $a_n \leq A$ for all values of n. Similarly, if there exists a finite number A with the property that $a_n \geq B$ for all values of n, then the sequence is said to be bounded below. The sequence is said to be bounded if it is either bounded above or bounded below.

The upper and lower limits of a sequence:

Let us consider the sequence $\{a_n\}$. If A is a number such that $a_n < A, n \geq N$, then A is called an inferior number for the sequence $\{a_n\}$.

Obviously, in this way every number less than A is an inferior number and there may exists number greater than A possessing the same property. The inferior numbers are unlimited and form an aggregate. If the upper bound of this aggregate of inferior numbers be λ,then λ is called the lower limit of $\{a_n\}$ and we write this as$\underline{lim} = \lambda$.

Similarly, B is number such that $B > a_n, n \geq N$, then A is called a superior number for the sequence $\{a_n\}$.

Obviously, in this way every number greater than B is a superior number and there may exists number less than B possessing the same property. The superior numbers are unlimited and form an aggregate. If the lower bound of this aggregate of superior number be μ, then μ is called the lower limit of $\{a_n\}$ and we write this as $\overline{lim} = \mu$.

It is easy to see that "If there exist no inferior numbers, then $\lambda = -\infty$ and if there exist no superior numbers, then $\mu = \infty$".

Example:

(i) Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$.. Here 1 is the least upper bound and 0 is the greatest lower bound. So it is a bounded sequence.

(ii) Consider the sequence $1,2,3, \ldots n, \ldots$ is bounded below but not bounded above. Here 1 is the greatest lower bound(G.L.B).

(iii) The sequence $-1, -2, -3, \ldots -n, \ldots$ is bounded above but not bounded below. Here -1 is the least upper bound(L.U.B).

(iv) The sequence $1, -1, 1, -1, \ldots$. is a bounded sequence, since -1 is the G.L.B and 1 is the L.U.B.

Theorem 6:

The necessary and sufficient condition for the convergence of $\{a_n\}$ is that $\mu = \lambda$. i.e., $\overline{lim}\, a_n = \underline{lim}\, a_n$.

Proof:

(i) **The condition is necessary**

Suppose that $a_n \to 1$, then by definition $|a_n - l| < \epsilon$ for all $n \geq N$.

$$\Rightarrow l - \epsilon < a_n < l + \epsilon \text{ for all } n \geq N$$

$l + \epsilon$ is a superior number and $l + \epsilon$ is an inferior number. The difference between these two is 2ϵ and can be made indefinitely small. Further no superior number can be less than an inferior number. Hence the lower bound of the superior numbers and the upper bound of the inferior numbers must coincide with l.

$$\text{i.e., } \overline{lim}\, a_n = \underline{lim}\, a_n.$$

This prove that the condition is necessary.

(ii) **The condition is sufficient.**

Suppose now that we are given$\overline{lim}\, a_n = \underline{lim}\, a_n = l$. Then since $\overline{lim}\, a_n$ is the lower bound of all the superior numbers.

$$\overline{lim}\, a_n + \epsilon$$

i.e., $l + \epsilon$ is the superior number.

$$\therefore a_n < l + \epsilon, \forall\, n \geq N_1$$

Similarly, $\underline{lim}\, a_n$ is the upper bound of all the inferior numbers.

$$\underline{lim}\, a_n + \epsilon$$

i.e., $l-\epsilon$ is an inferior number.

$$\therefore l-\epsilon < a_n, \forall\, n \geq N_2$$

If N is greater than N_1 and N_2, then we have, $l-\epsilon < a_n < l+\epsilon$ for all $n \geq N$ and $|a_n - l| < \epsilon$ for all $n \geq N$. Hence $a_n \to 1$.

This prove that the condition is sufficient.

Theorem 7:

Cauchy's general principle of convergence:

A necessary and sufficient condition for existence of a limit to the sequence $\{a_n\}$ is that, if any positive integer ϵ has been chosen, as small as we please, there shall be a positive number m such that $|a_n - a_m| < \epsilon$ for all $n \geq m$.

Proof:

(i) **The condition is necessary.**

Let the sequence converge to the limit l. Having to choose ϵ, take it as ${}^{\epsilon}/_{2}$.

We know that there is a positive integer m such that

$$|a_n - l| < {}^{\epsilon}/_{2} \text{ for all } n \geq m.$$

But $a_n - a_m = a_n - l + l - a_m$

$$\Rightarrow |a_n - a_m| \leq |a_n - l| + |l - a_m| < {}^{\epsilon}/_{2} + {}^{\epsilon}/_{2} < \epsilon, if\ n \geq m.$$

(ii) **The condition is sufficient.**

For if the condition is satisfied, then there exists an integer m such that

$$|a_n - a_m| < \epsilon \text{ for all } n \geq m.$$

$$i.e., a_m - \epsilon < a_n < a_m + \epsilon \text{ for all } n \geq m.$$

$a_m - \epsilon$ is an inferior number and $a_m + \epsilon$ is the superior number.

Hence $\mu - \lambda \leq (a_m + \epsilon) - (a_m + \epsilon) \leq 2\epsilon$

i.e., $\mu - \lambda \leq 2\epsilon$.

But $\mu - \lambda \geq 0$ and ϵ can be taken arbitrarily small.

$$\therefore \mu - \lambda = 0 \Rightarrow \mu = \lambda.$$

Thus $\overline{lim}\, a_n = \underline{lim}\, a_n$.

Hence the sequence $\{a_n\}$is convergent.

Monotonic sequence:

A sequence in which $a_{n+1} \geq a_n$for all values of n is called a monotonic increasing sequence. Similarly, if $a_{n+1} \leq a_n$ for all values of n is called a monotonic decreasing sequence.

Example:

(i) $\{a_n\}$ defined by $a_n = n$ is monotonically increasing

(ii) $\{a_n\}$ defined by $a_n = {}^{1}/_{n}$ is monotonically decreasing.

(iii) $\{a_n\}$ defined by $a_n = (-1)^n$ is neither monotonically increasing or monotonically decreasing.

Problems:

(1) Show that $\left\{\frac{n}{n+1}\right\}$ is a monotonic increasing sequence.

Solution:

Given: $\{a_n\} = \left\{\frac{n}{n+1}\right\}$

Let $a_n = \frac{n}{n+1}$. Then $a_{n+1} = \frac{n+1}{n+2}$.

Now,

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1}$$
$$= \frac{(n+1)^2 - n(n+2)}{(n+1)(n+2)}$$
$$= \frac{1}{(n+1)(n+2)} > 0.$$

Hence $a_{n+1} > a_n$,for all n.

Hence showed.

(2) Prove that $\left\{\frac{2n-7}{3n+2}\right\}$ is a monotonic increasing sequence.

Solution:

Given: $\{a_n\} = \left\{\frac{2n-7}{3n+2}\right\}$.

Let $a_n = \frac{2n-7}{3n+2}$.

Then $a_{n+1} = \frac{2(n+1)-7}{3(n+1)+2}$

$\Rightarrow a_{n+1} = \frac{2n-5}{3n+5}$.

Now,

$$a_{n+1} - a_n = \frac{2n-5}{3n+5} - \frac{2n-7}{3n+2}$$
$$= \frac{(2n-5)(3n+2) - (2n-7)(3n+5)}{(3n+5)(3n+2)}$$
$$= \frac{25}{(3n+5)(3n+2)} > 0.$$

Hence proved.

Theorem 8:

A monotonic sequence always tends to a limit finite or infinite.

Proof:

Suppose that the sequence $\{a_n\}$ is a monotonic increasing sequence.

Now, all the terms of the sequence $a_1, a_2, a_3 \dots, a_n, \dots$ form an aggregate. If this aggregate be not bounded above, its is clear that terms continuously increase and tend to plus infinity. If it is bounded above, let M be the upper bound.

Then we have $a_n < M$ for all n and $a_n > M - \epsilon$ for at least one value of n, say N so that $a_N > M - \epsilon$.

But as the sequence is steadily increasing, we have

$a_n > M - \epsilon$ for at least one value of $n \geq N$

$\Rightarrow |a_n - M| < \epsilon$ for at least one value of $n \geq N$

$\therefore \{a_n\} \to M$.

Note:

(i) Thus we have a monotonic increasing if bounded above tends to the upper bound and if not bounded above tends to $+\infty$.

(ii) Similarly, a monotonic decreasing if bounded below tends to its lower bound and if not bounded below tends to $-\infty$.

Examine whether the following sequence are monotonic:

(a) $\left(2+\frac{1}{n}\right)$ **(b)**$\left({}^{1}\!/_{2^n}\right)$ **(c)** $\left(\frac{1}{n!}\right)$ **(d)** $\left(\frac{(-1)^n}{n}\right)$

(a) $\left(2+\frac{1}{n}\right)$

Given: $\{a_n\} = \left(2+\frac{1}{n}\right)$

Let $a_n = 2+\frac{1}{n}$.

Then$a_{n+1} = 2+\frac{1}{n+1}$

Now

$$a_{n+1} - a_n = 2 + \frac{1}{n+1} - 2 + \frac{1}{n} = \frac{1}{n+1} - + \frac{1}{n} = \frac{-1}{n(n+1)} < 0.$$

$\therefore a_{n+1} < a_n$, for all values of n.

Hence given sequence is monotonic decreasing.

(b) $\left({}^{1}\!/_{2^n}\right)$

Given: $\{a_n\} = \left({}^{1}\!/_{2^n}\right)$

Let $a_n = {}^{1}\!/_{2^n}$.

Then $a_{n+1} = {}^{1}\!/_{2^{n+1}}$

Now

$$a_{n+1} - a_n = {}^{1}\!/_{2^{n+1}} - {}^{1}\!/_{2^n} = \frac{1-2}{2^{n+1}}$$

$$\Rightarrow a_{n+1} - a_n = \frac{-1}{2^{n+1}} < 0$$

$\Rightarrow a_{n+1} < a_n$, for all values of n.

Hence given sequence is monotonic decreasing.

(c)$\left(\frac{1}{n!}\right)$

Given: $\{a_n\} = \left(\frac{1}{n!}\right)$

Let $a_n = \frac{1}{n!}$.Then $a_{n+1} = \frac{1}{(n+1)!}$

Now

$$a_{n+1} - a_n = \frac{1}{(n+1)!} - \frac{1}{n!}$$

$$a_{n+1} - a_n = \frac{1}{(n+1)n!} - \frac{1}{n!} = \frac{1-(n+1)}{n!\,(n+1)} = \frac{-n}{(n+1)!} < 0.$$

$\Rightarrow a_{n+1} < a_n$, for all values of n.

Hence given sequence is monotonic decreasing.

(d) $\left(\frac{(-1)^n}{n}\right)$

Given: $\{a_n\} = \left(\frac{(-1)^n}{n}\right)$

Let $a_n = \frac{(-1)^n}{n}$.Then $a_{n+1} = \frac{(-1)^{n+1}}{n+1}$

Now

$$a_{n+1} - a_n = \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^n}{n}$$

$$a_{n+1} - a_n = (-1)^n \left[\frac{1}{n+1} - \frac{1}{n}\right] = (-1)^n \left[\frac{-1}{n(n+1)}\right] < 0.$$

$\Rightarrow a_{n+1} < a_n$, for all values of n.

Hence given sequence is monotonic decreasing.

Problems:

(1) Let $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}$. Show that the sequence $\{a_n\}$ tends to a limit.

Solution:

Given $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}$

$$a_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + \cdots + \frac{1}{2n+2}$$

Now

$$a_{n+1} - a_n = \left[\frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + \cdots + \frac{1}{2n+2}\right] - \left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}\right]$$

$$= \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} - \frac{1}{n+2}$$

$$- \frac{1}{n+3} - \cdots - \frac{1}{2n} = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2n+1} + \frac{1}{2(n+1)} - \frac{1}{n+1}$$

$$= \frac{1}{2n+1} + \frac{1-2}{2(n+1)} = \frac{1}{2n+1} - \frac{1}{2(n+1)}$$

$$= \frac{1}{2(n+1)(2n+1)} > 0$$

$\therefore \{a_n\}$ is a monotonic increasing sequence.

To show that this sequence tends to a limit, it is necessary o show that $\{a_n\}$is bounded above.

Here

$$a_n < \frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+1} = n\left(\frac{1}{n+1}\right) \Rightarrow a_n < 1.$$

Hence (a_n) is a bounded monotonically increasing sequence and so it tends to a limit.

(2)Find the limit of the sequence $\{a_n\}$ where $\{a_n\} = \left(1 + \left(\frac{1}{n}\right)^n\right)$?

Solution:

Given: $\left(1 + \left(\frac{1}{n}\right)^n\right)$

By binomial theorem,

$$(1+x)^n = 1^n + nC_1 x + nC_2 x^2 + \cdots + nC_n x^n.$$

$$\left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots$$

$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{(n-)}{n}\right).$$

This expression contains (n+1) terms. As n increases, the number of terms also increases. Hence $\{a_n\}$ is a monotonically increasing sequence.

Hence either it tends to limit or infinity. Thus in order to show that the expression tends to a limit, it is necessary to show that it is bounded.

i.e.,

$$\left(1+\frac{1}{n}\right)^n < 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$$
$$< 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}$$
$$< 1+\frac{1-\left({}^{1}\!/_{2}\right)^n}{1-{}^{1}\!/_{2}}$$
$$< 1+\frac{2^n-1}{2^{n-1}}$$
$$< 1+2-\frac{1}{2^{n-1}}$$
$$\Rightarrow \left(1+\frac{1}{n}\right)^n < 3-\frac{1}{2^{n-1}}$$
$$\Rightarrow \left(1+\frac{1}{n}\right)^n < 3$$

Hence $\left(1+\frac{1}{n}\right)^n$ is bounded and tends to a limit. This limit is denoted by e. Hence

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Clearly

$$\left(1+\frac{1}{n}\right)^n > 2 \Rightarrow 2 < e < 3.$$

(3) Let $a_{n+1}=\frac{1}{2}[a_n+b_n], b_{n+1}=\sqrt{a_nb_n}$. Show that the sequences $(a_n)\&(b_n)$ converge to a common limit.

Solution:

By hypothesis, a_{n+1} and $b_{n+1}$are respectively the $A.M$ & $G.M$ between $(a_n)\&(b_n)$. Also we know that $A.M$ & $\geq G.M$. Hence

$$a_{n+1} \geq b_{n+1}. \qquad \text{--------(1)}$$

Moreover the $A.M$ & $G.M$ of two number lie between the numbers.

$$a_n \geq a_{n+1} \geq b_n, \forall\, n \in \mathbb{N}. \text{--------(2)}$$
$$a_n \geq b_{n+1} \geq b_n, \forall\, n \in \mathbb{N}. \qquad \text{--------(3)}$$
$$\therefore a_n \geq a_{n+1} \geq b_{n+1} \geq b_n, \forall\, n \in \mathbb{N} \qquad \text{(by (2) and (3))}$$

Hence (a_n) is a monotonically decreasing sequence and $(b_n)$is a monotonically increasing sequence.

Further

$$a_n \geq b_n \geq b_1, \forall\, n \in \mathbb{N}\ \& b_n \leq a_n \leq a_1, \forall\, n \in \mathbb{N}$$

Hence (a_n) is a monotonically decreasing sequence and bounded below by b_1.
Similarly, $(b_n)$is a monotonically increasing sequence and bounded above by a_1.
Thus $(a_n) \to l(say)$ and $(b_n) \to m(say)$.
Now,

$$a_{n+1}=\frac{1}{2}(a_n+b_n).$$

Taking limit as $n\to\infty$, we get,

$$l=\frac{1}{2}(l+m) \Rightarrow l-\frac{l}{2}=\frac{m}{2} \Rightarrow \frac{l}{2}=\frac{m}{2} \Rightarrow l=m.$$

(4)Let $a_{n+1} = \sqrt{a_{n+1}a_n}$ and $a_n > 0$. Show that the sequences $\{a_{2n} - 1\}$&$\{a_{2n}\}$ are both monotonic, one decreasing and other increasing. Also prove that $\{a_n\}$ tends to $(a_1 a_2{}^2)^{1/3}$.

Solution:

Let $a_1 > a_2$; $a_3 = \sqrt{a_1 a_2}$; $a_4 = \sqrt{a_3 a_2}$.

a_3 lies between a_2& a_1 and a_4 lies between a_3& a_2.

From this we see that $a_2, a_4, a_6, \ldots$ is a monotonic increasing sequence and $a_1, a_3, a_5, \ldots$ is a monotonic decreasing sequence.

Now

$$a_3{}^2 = a_1 a_2 \Rightarrow \frac{a_3{}^2}{a_2{}^2} = \frac{a_1}{a_2} \Rightarrow \frac{a_3}{a_2} = \left(\frac{a_1}{a_2}\right)^{1/2};$$

$$a_4{}^2 = a_3 a_2 \Rightarrow \frac{a_4{}^2}{a_2{}^2} = \frac{a_3}{a_2} = \left(\frac{a_1}{a_2}\right)^{1/2};$$

$$a_5{}^2 = a_4 a_3 \Rightarrow \frac{a_5{}^2}{a_2{}^2} = \frac{a_4}{a_2}.\frac{a_3}{a_2} = \left(\frac{a_1}{a_2}\right)^{1/2+1/4} = \left(\frac{a_1}{a_2}\right)^{3/4};$$

$$a_6{}^2 = a_5 a_4 \Rightarrow \frac{a_6{}^2}{a_2{}^2} = \frac{a_5}{a_2}.\frac{a_4}{a_2} = \left(\frac{a_1}{a_2}\right)^{3/8}.\left(\frac{a_1}{a_2}\right)^{1/4} = \left(\frac{a_1}{a_2}\right)^{3/8+1/4} = \left(\frac{a_1}{a_2}\right)^{5/16}.$$

Continuing this process, we get,

$\frac{a_{n+2}}{a_2} = \left(\frac{a_1}{a_2}\right)^{u_n}$, where u_n is the n^{th} term of the sequence $1/2, 1/4, 3/8, 5/16, \ldots$.

Here

$$u_n = \frac{1}{2}(u_{n-1} + u_{n-2})$$

$$u_n - u_{n-1} = \left(-\frac{1}{2}\right)(u_{n-1} - u_{n-2})$$

$$u_{n-1} - u_{n-2} = \left(-\frac{1}{2}\right)(u_{n-2} - u_{n-3})$$

$$\ldots.$$

$$\ldots.$$

$$u_3 - u_2 = \left(-\frac{1}{2}\right)(u_2 - u_1)$$

$$u_n - u_{n-1} = \left(-\frac{1}{2}\right)^{n-2}(u_2 - u_1) = \left(-\frac{1}{2}\right)^{n-2}(1/4 - 1/2) = (-1)^{n-1}\left(\frac{1}{2^n}\right).$$

Hence

$$u_n - u_{n-1} = (-1)^{n-1}\left(\frac{1}{2^n}\right)$$

$$u_{n-1} - u_{n-2} = (-1)^{n-2}\left(\frac{1}{2^{n-1}}\right)$$

$$\ldots.$$

$$\ldots.$$

$$u_2 - u_1 = (-1)^1\left(\frac{1}{2^2}\right).$$

Thus

$$u_n - u_1 = -\frac{1}{2^2} + \frac{1}{2^3} - \cdots + (-1)^{n-1}\left(\frac{1}{2^n}\right)$$

$$u_n = \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \cdots = \frac{1}{3}\left(\frac{1-\left(-\frac{1}{2}\right)^n}{1-\left(-\frac{1}{2}\right)}\right)$$

$$\therefore \lim_{n\to\infty} u_n = \frac{1}{3}.$$

$$\lim_{n\to\infty} \frac{a_{n+2}}{a_2} = \left(\frac{a_1}{a_2}\right)^{\frac{1}{3}} \Rightarrow \lim_{n\to\infty} a_{n+2} = a_2\left(\frac{a_1}{a_2}\right)^{\frac{1}{3}} \Rightarrow \lim_{n\to\infty} a_{n+2} = (a_1 a_2{}^2)^{\frac{1}{3}}.$$

Series

Infinite series:

Let $u_r$be a function of r which has a definite value for all integral values r. An expression of the form

$$u_1 + u_2 + u_{3+} \dots + u_n + \cdots$$

In which every term is followed by another term called an infinite series. This series is denoted by $\sum_{r=1}^{\infty} u_r$and the sum of the first n terms of series namely $u_1 + u_2 + u_{3+} \dots + u_n$ by S_n.

1) S_n may tend to a finite limit(say)
2) S_n may tend to infinity
3) S_n may tend to minus infinity
4) S_n may tend to more than one limit

If S_n tends to a finite limit S, then the series is said to be convergent and S is called its infinity.

The sum to infinity is not a sum in the ordinary senses, but it is a limit of a sum.

Consider the series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

$$S_n = \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}}$$

Now as $n \to \infty, \frac{1}{2^{n-1}} \to 0$. Hence $\lim_{n\to\infty} S_n = 2$.

Therefore, the series is convergent.

If S_n tends to infinity or mins infinity, then the series is said to be divergent.

Let us consider the series $\sum_{r=1}^{\infty} u_r$ where $u_r = r$.

$$S_n = 1 + 2 + 3 + \cdots + n = \frac{1}{2} n(n+1) \Rightarrow S_n \to \infty, \text{when } n \to \infty.$$

$$\Rightarrow \sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty} r \text{ is divergent.}$$

If S_n tends to more than one limit, then the series is said to be oscillate. In this case, we say that the series oscillates finitely or infinitely according as S_n oscillates between finite limits or between $+\infty$ and $-\infty$.

Consider the series $\sum_{r=1}^{\infty} (-1)^{r+1}$.

$$S_n = 1 - 1 + 1 - 1 + \cdots + n \text{ terms.}$$

$$\therefore S_n = 1, \text{if n is odd and } S_n = 0, \text{if n is even.}$$

$$\sum_{r=1}^{\infty} (-1)^{r+1} \text{ oscillates finitely.}$$

Consider the series $\sum_{r=1}^{\infty} (-1)^{r+1} r$.

$$S_n = 1 - 2 + 3 - 4 + \cdots + n \text{ terms.}$$

$$\therefore S_n = \frac{1}{2}n, \text{if n is even and } S_n = \frac{1}{2}(n+1), \text{if n is odd.}$$

or

$$S_n \to -\infty, \text{if n is even and } S_n \to +\infty, \text{if n is odd, as n} \to \infty.$$

$$\sum_{r=1}^{\infty}(-1)^{r+1}r \text{ oscillates infinitely.}$$

If the series is divergent or oscillating, it does not posses "sum to infinity" as defined above.

Consider the geometric series $1 + x + x^2 + \cdots + x^{n-1} + \cdots$

$$S_n = \frac{1 - x^n}{1 - x}, \text{if } x \neq 1.$$

If $|x| < 1. \text{i.e.}, -1 < x < 1, x^n \to 0.$

$$S_n \to \frac{1}{1-x}, \text{as n} \to \infty.$$

If $|x| \geq 1, S_n \to \infty.$

If $|x| = 1, S_n = 1$ or 0 according as n is odd or even.

$$\therefore \text{the geometric series} \sum_{n=0}^{\infty} x^n = \begin{cases} \text{converges, if } |x| < 1 \\ \text{diverges if } |x| \geq 1 \\ \text{oscillates finitely if x} = -1 \\ \text{infinitely if x} < -1 \end{cases}.$$

Note:

If the sum to n terms, then S_n can be expressed by elementary functions, the nature of the series can be determined by finding whether the expression for S_n tends to a limit or diverges or oscillates when $n \to \infty$. But there are cases in which we have no method to find the sum of the first n terms of a series. So we have to find methods for deciding the question of convergence when it is impossible or inconvenient to find S_n in this way.

Consider the geometric series $1 + x + x^2 + \cdots + x^{n-1} + \cdots$

$$S_n = \frac{1 - x^n}{1 - x}, \text{if } x \neq 1.$$

If $|x| < 1. \text{i.e.}, -1 < x < 1, x^n \to 0.$

$$S_n \to \frac{1}{1-x}, \text{as n} \to \infty.$$

If $|x| \geq 1, S_n \to \infty.$

If $|x| = 1, S_n = 1$ or 0 according as n is odd or even.

If $|x| < -1, S_n \to \infty$ or $-\infty$ according as n is odd or even.

$$\therefore \text{the geometric series} \sum_{n=0}^{\infty} x^n = \begin{cases} \text{converges, if } |x| < 1 \\ \text{diverges if } |x| \geq 1 \\ \text{oscillates finitely if x} = -1 \\ \text{infinitely if } 1 + \text{x} < 0 \end{cases}.$$

Theorem:1

If $u_1 + u_2 + u_{3+} \ldots + u_n + \cdots$ is convergent and has the sum "s", then $u_{m+1} + u_{m+2} + u_{m+3} + \cdots$ is convergent and has the sum $(u_1 + u_2 + u_{3+} \ldots + u_m)$ where m is any positive integer.

Proof:

Let $u_1 + u_2 + u_{3+} \dots + u_n + \cdots$ be convergent.

Given:

$$\lim_{n\to\infty}(u_1 + u_2 + u_{3+} \dots + u_m) = s$$

Let $u_{m+1} + u_{m+2} + u_{m+3} + \cdots + u_{m+n} = (u_{m+1} + u_{m+2} + u_{m+3} + \cdots u_m + \cdots + u_{m+n}) - (u_1 + u_2 + u_{3+} \dots + u_m)$

Taking limit on both sides, we get,

$$\lim_{n\to\infty}(u_{m+1} + u_{m+2} + u_{m+3} + \cdots + u_{m+n})$$
$$= \lim_{n\to\infty}(u_{m+1} + u_{m+2} + u_{m+3} + \cdots u_m + \cdots + u_{m+n})$$
$$- \lim_{n\to\infty}(u_1 + u_2 + u_{3+} \dots + u_m) = S - (u_1 + u_2 + u_{3+} \dots + u_m).$$

Similarly, if $u_1 + u_2 + u_{3+} \dots + u_n + \cdots$ diverges, then $u_{m+1} + u_{m+2} + u_{m+3} + \cdots$ diverges, where m is given any positive integer.

Note:

The convergence, the divergence and oscillation series is not affected by the addition, omission or alteration of a finite no. of terms.

Theorem:2

If $u_1 + u_2 + u_{3+} \dots + u_n + \cdots$ is convergent and has the sum "s", then $(ku_1 + ku_2 + ku_{3+} \dots + ku_n + \cdots)$ and has the sum "ks".

Proof:

Let $u_1 + u_2 + u_{3+} \dots + u_n + \cdots$ be convergent.

Given:

$$\lim_{n\to\infty}(u_1 + u_2 + u_{3+} \dots + u_n) = s$$

$$\lim_{n\to\infty}(ku_1 + ku_2 + ku_{3+} \dots + ku_n + \cdots)$$
$$= \lim_{n\to\infty} k(u_1 + u_2 + u_{3+} \dots + u_n) = k\lim_{n\to\infty}(u_1 + u_2 + u_{3+} \dots + u_n)$$
$$\lim_{n\to\infty}(ku_1 + ku_2 + ku_{3+} \dots + ku_n) = ks$$

Hence $(ku_1 + ku_2 + ku_{3+} \dots + ku_n + \cdots)$ converges to ks.

Theorem:3

If $u_1 + u_2 + u_{3+} \dots + u_n + \cdots$ and $v_1 + v_2 + v_{3+} \dots + v_n + \cdots$ are both convergent, then the series $\sum(u_n + v_n)$ is convergent and its sum is the sum of the two series.

Proof:

Let $u_1 + u_2 + u_{3+} \dots + u_n + \cdots$ and $v_1 + v_2 + v_{3+} \dots + v_n + \cdots$ be two convergent series. Let the sum of the two series be "s" and "y" respectively. Then

$$\lim_{n\to\infty}\sum \boldsymbol{u_n} = s \text{ and } \lim_{n\to\infty}\sum \boldsymbol{v_n} = t$$

$$\therefore \lim_{n\to\infty}\sum(u_n + v_n) = (u_1 + v_1) + (u_2 + v_2) + \cdots + (u_n + v_n)$$

Taking limit on both sides, we get,

$$\lim_{n\to\infty}\sum(u_n + v_n) = \lim_{n\to\infty}[(u_1 + v_1) + (u_2 + v_2) + \cdots + (u_n + v_n)]$$
$$= \lim_{n\to\infty}(u_1 + u_2 + u_{3+} \dots + u_n) + \lim_{n\to\infty}(v_1 + v_2 + v_{3+} \dots + v_n)$$
$$= \lim_{n\to\infty}\sum(u_n + v_n) = s + t$$

$$\sum(u_n + v_n) \to (s+t).$$

Series of positive terms:

Theorem: 4

A series of positive terms cannot oscillate is either convergent or divergent.

Proof:

Let us consider a positive term.

Since all the terms are positive, S_n steadily increases as n increases. It tends to a finite limit(or) infinity. Hence the series cannot oscillate.

If $S_n < k$, for all values of n, $\lim_{n\to\infty} S_n$ exists and is equal to "k" or is less than the "k". Then the series is convergent.

Theorem:5

If $u_1 + u_2 + u_{3+} \dots + u_n + \cdots$ is convergent, then $\lim_{n\to\infty} u_n = 0$.

Proof:

Let $u_1 + u_2 + u_{3+} \dots + u_n + \cdots$ be a convergent series.

Since the series is convergent,

$$\lim_{n\to\infty}(u_1 + u_2 + u_{3+} \dots + u_n) = s$$

Now,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}(u_1 + u_2 + u_{3+} \dots + u_n) - \lim_{n\to\infty}(u_1 + u_2 + u_{3+} \dots + u_{n-1}) = s - s = 0.$$

$$\Rightarrow \lim_{n\to\infty} u_n = 0.$$

Hence proved.

Comparison test:

If $u_1 + u_2 + u_{3+} \dots + u_n + \cdots$ and $v_1 + v_2 + v_{3+} \dots + v_n + \cdots$ are two series of positive terms and the second series is convergent and $u_n \le kv_n$, where k is constant for all values of "n", then the first series is also converges and its sum is less than or equal to "k" times that of all the second.

Conversely, if $\sum u_n$ is divergent and $u_n \ge kv_n$, then $\sum v_n$ is divergent.

Problems:

1) **Prove that $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$ is convergent.**

Proof:

Given: $\sum u_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

Here $u_n = \frac{1}{(n-1)!}$. Take $v_n = \frac{1}{n!} = \frac{1}{1.2.3\dots n} < \frac{1}{1.2.2.2\dots 2} < \frac{1}{2^{n-1}}$.

Since $\frac{1}{2^{n-1}}$ is convergent, $\sum u_n$ is convergent.

2) **Prove that the general harmonic series $\frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \cdots$ is divergent, where "a" and "b" are the positive numbers diverges to infinity.**

Solution:

Given: $\sum u_n = \frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \cdots$

Here $u_n = \frac{1}{a+(n-1)b}$. Take $v_n = \frac{1}{a+nb}$.

$$u_{n+1} = \frac{1}{a+nb} > \frac{1}{n(a+b)}.$$

Since $\frac{1}{n}$ is divergent and $u_{n+1} > \frac{1}{n(a+b)}$, $\sum u_n$ is divergent.

3) Prove that the series $\frac{1}{1.3} + \frac{2}{3.5} + \frac{3}{5.7} + \cdots$ is divergent.

Proof:

Given: $\sum u_n = \frac{1}{1.3} + \frac{2}{3.5} + \frac{3}{5.7} + \cdots = \frac{n}{(2n-1)(2n+1)} > \frac{1}{6n}$

Since $\frac{1}{n}$ is divergent and $\sum u_n > \frac{1}{6n}$, $\sum u_n$ is divergent.

Theorem:6

If $\sum v_n$ is convergent and $\frac{u_n}{v_n}$ tends to a limit other than zero as $n \to \infty$, then $\sum u_n$ is convergent.

Proof:

Let $\lim_{n\to\infty} \frac{u_n}{v_n}$ be k. (k being positive$\neq 0$.)

On and after a certain value of "n" say "m" the values of the series of numbers.

$\frac{u_n}{v_n}, \frac{u_{n+1}}{v_{n+1}}, \frac{u_{n+2}}{v_{n+2}}, \ldots$ lie in the interval $k-\epsilon$ to $k+\epsilon$ where "ϵ" is a very small finite the quantity.

As $\lim_{n\to\infty} \frac{u_n}{v_n} = k$, $\left|\frac{u_n}{v_n} - k\right| < \epsilon, \forall\, n \geq m$, a positive integer.

$$k+\epsilon < \frac{u_n}{v_n} < k+\epsilon \Rightarrow \frac{u_n}{v_n} < k+\epsilon \Rightarrow u_n < v_n(k+\epsilon), \forall \text{ n} \geq \text{m}.$$

$\sum u_n$ is convergent.

Hence proved.

Theorem:7

If $\sum v_n$ is divergent and $\frac{u_n}{v_n}$ tends to a limit other than zero as $n \to \infty$, then $\sum u_n$ is divergent.

Proof:

Let $\sum v_n$ is divergent.

Let $\lim_{n\to\infty} \frac{u_n}{v_n}$ be k. (k being positive$\neq 0$.)

As $\lim_{n\to\infty} \frac{u_n}{v_n} = k$, $\left|\frac{u_n}{v_n} - k\right| < \epsilon, \forall\, n \geq N$, a positive integer.

$$k-\epsilon < \frac{u_n}{v_n} < k-\epsilon \Rightarrow k+\epsilon < \frac{u_n}{v_n} \Rightarrow v_n(k-\epsilon) < u_n \Rightarrow u_n > v_n(k-\epsilon), \forall \text{ n} \geq \text{m}.$$

$\sum v_n$ is divergent.

Hence proved.

Problems:

1) Test the convergence of $\sum \frac{1}{\sqrt{n^2+1}}$.

Proof:

Given: $\sum u_n = \sum \frac{1}{\sqrt{n^2+1}}$.

Take $\sum v_n = \sum \frac{1}{n}$ which is divergent.

$$\therefore \frac{u_n}{v_n} = \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = \frac{n}{\sqrt{n^2+1}} = \frac{n}{n\sqrt{1+\frac{1}{n^2}}} \Rightarrow \frac{u_n}{v_n} = \frac{1}{\sqrt{1+\frac{1}{n^2}}} \Rightarrow \lim \frac{u_n}{v_n} = 1 \neq 0$$

Thus $\sum u_n$ and $\sum v_n$ may be both converge or diverge.

Also $\sum \frac{1}{n}$ is divergent.

Hence $\sum u_n$ is also divergent.

2) **Test the convergence of $\frac{1}{1.2.3} + \frac{2}{2.3.5} + \frac{3}{3.4.5} + \cdots$**

Proof:

Given: $\sum u_n = \frac{1}{1.2.3} + \frac{2}{2.3.5} + \frac{3}{3.4.5} + \cdots$

Now $u_n = \frac{2n-1}{n(n+1)(n+2)}$

Take $v_n = \frac{1}{n^2}$. And we know that $\sum v_n$is convergent

$$\therefore \frac{u_n}{v_n} = \frac{2n-1}{n(n+1)(n+2)}.n^2 = \frac{n(2-\frac{1}{n})}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 2 \neq 0$$

Thus $\sum u_n$ and $\sum v_n$ may be both converge or diverge.

Also $\sum \frac{1}{n^2}$ is convergent.

Hence $\sum u_n$ is also convergent.

3) **Test the convergence or divergence of the series $\sum_1^\infty\left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)$**

Proof:

Given: $\sum u_n = \sum_1^\infty\left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)$

Let $u_n = \sqrt{n^4+1} - \sqrt{n^4-1} = \sqrt{n^4+1} - \sqrt{n^4-1}.\frac{\sqrt{n^4+1}+\sqrt{n^4-1}}{\sqrt{n^4+1}+\sqrt{n^4-1}}$

$$u_n = \frac{n^4+1-(n^4-1)}{\left(\sqrt{n^4+1}+\sqrt{n^4-1}\right)} = \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

Let $\sum v_n = \sum \frac{1}{n^2}$ & $\sum v_n$ is convergent.

$$\therefore \frac{u_n}{v_n} = \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}.n^2 \Rightarrow \frac{u_n}{v_n} = \frac{2n^2}{n^2\left(\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\right)}$$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2}{\left(\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\right)} = \frac{2}{2} = 1 \neq 0.$$

Thus $\sum u_n$ and $\sum v_n$ may be both converge or diverge.

Also $\sum \frac{1}{n^2}$ is convergent.

Hence $\sum u_n$ is also convergent.

4) **Discuss the convergence of the series $\sum_1^\infty \frac{1}{(a+n)^p(b+n)^q}$, a, b, p, q are all positive numbers.**

Proof:

Given: $\sum u_n = \sum_1^\infty \left(\frac{1}{(a+n)^p(b+n)^q}\right)$

Let $u_n = \frac{1}{(a+n)^p(b+n)^q}$

Take $\sum v_n = \sum \frac{1}{n^{p+q}}$

$$\therefore \frac{u_n}{v_n} = \frac{n^{p+q}}{(a+n)^p(b+n)^q} = \frac{n^{p+q}}{n^{p+q}\left(1+\frac{a}{n}\right)^p\left(1+\frac{b}{n}\right)^q}$$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{a}{n}\right)^p\left(+\frac{b}{n}\right)^q} = 1 \neq 0.$$

$\Rightarrow \sum u_n$ and $\sum v_n$ may be both converge or diverge. Also $\sum v_n$ is converges if $p+q>1$ and diverge if $p+q\leq 1$. Thus $\sum u_n$ is converges if $p+q>1$ and diverges if $p+q\leq 1$.

5) **Find whether series in which $u_n = (n^3+1)^{1/3} - n$ is convergent (or) divergent.**

Solution:

Given: $\sum u_n = \sum_1^\infty \left((n^3+1)^{1/3} - n\right)$

$$\text{Let } u_n = (n^3+1)^{1/3} - n = n\left[\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right] = n\left[1+\frac{1}{3}.\frac{1}{1!}.\frac{1}{n^3} - \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}.\frac{1}{(n^3)^2} + \cdots - 1\right] = n\left[\frac{1}{3n^3} - \frac{1}{9n^6} + \cdots\right]$$

$\Rightarrow u_n = \left[\frac{1}{3n^2} - \frac{1}{9n^5} + \cdots\right]$. Take $\sum v_n = \sum \frac{1}{n^2}$

$$\text{Now, } \frac{u_n}{v_n} = \frac{\left[\frac{1}{3n^2} - \frac{1}{9n^5} + \cdots\right]}{\frac{1}{n^2}}$$

$$= n^2\left[\frac{1}{3n^2} - \frac{1}{9n^5} + \cdots\right]$$

$$= \left[\frac{1}{3} - \frac{1}{9n^3} + \cdots\right]$$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left[\frac{1}{3} - \frac{1}{9n^3} + \cdots\right] = \frac{1}{3} \neq 0.$$

Thus $\sum u_n$ and $\sum v_n$ may be both converge or diverge.

Since $\sum \frac{1}{n^2}$ is convergent, $\sum u_n$ is also convergent.

D'Alembert's Ratio Test

Conditions:

(i) **If $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = k$, where $k<1$, then $\sum u_n$ is convergent.**

(ii) **If $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = k$, where $k>1$, then $\sum u_n$ is divergent.**

(iii) **If $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = 1$, then the test fails.**

Problems:

1) **Test for convergence the series $\sum_{n=0}^\infty \frac{n^3+1}{2^n+1}$.**

Solution:

Given: $\sum u_n = \sum_{n=0}^{\infty} \frac{n^3+1}{2^n+1}$

Let $u_n = \frac{n^3+1}{2^n+1}$.

Then $u_{n+1} = \frac{(n+1)^3+1}{2^{n+1}+1} = \frac{n^3+3n^2+3n+2}{2^{n+1}+1}$.

$$\therefore \frac{u_{n+1}}{u_n} = \frac{n^3+3n^2+3n+2}{2^{n+1}+1} \cdot \frac{2^n+1}{n^3+1} = \frac{n^3\left(1+\frac{3}{n}+\frac{3}{n^2}+\frac{2}{n^3}\right)}{2^n(2+\frac{1}{2^n})} \cdot \frac{2^n(2+\frac{1}{2^n})}{n^3\left(1+\frac{1}{n^3}\right)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\left(1+\frac{3}{n}+\frac{3}{n^2}+\frac{2}{n^3}\right)}{(2+\frac{1}{2^n})} \cdot \frac{(2+\frac{1}{2^n})}{\left(1+\frac{1}{n^3}\right)} = \frac{1}{2} < 1.$$

Hence $\sum u_n$ is also convergent.

2) Examine the convergence of the series $\frac{1}{1^k} + \frac{x}{3^k} + \frac{x^2}{5^k} + \cdots + \frac{x^{n-1}}{(2n-1)^k} + \cdots$

Solution:

Given: $\sum u_n = \frac{1}{1^k} + \frac{x}{3^k} + \frac{x^2}{5^k} + \cdots + \frac{x^{n-1}}{(2n-1)^k} + \cdots$

Let $u_n = \frac{x^{n-1}}{(2n-1)^k}$.

Then $u_{n+1} = \frac{x^n}{(2n+1)^k}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{x^n}{(2n+1)^k} \cdot \frac{(2n-1)^{k^{n-1}}}{x^{n-1}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = x.$$

By D'Alembert's Ratio Test,

(i) If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = k$, where $k < 1$, then $\sum u_n$ is convergent.

(ii) If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = k$, where $k > 1$, then $\sum u_n$ is divergent.

(iii) If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, then the test fails.

In this condition,

$$\sum u_n = \frac{1}{1^k} + \frac{x}{3^k} + \frac{x^2}{5^k} + \cdots + \frac{x^{n-1}}{(2n-1)^k} + \cdots$$

Now,

$$u_n = \frac{x^{n-1}}{(2n-1)^k}$$

Take $\sum v_n = \sum \frac{1}{n^k}$

$$\frac{u_n}{v_n} = \frac{n^k}{(2n-1)^k} = \frac{1}{2^k(1-\frac{1}{2n})^k}$$

$$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{2^k(1-\frac{1}{2n})^k} = \frac{1}{2^k} \neq 0$$

Thus $\sum u_n$ and $\sum v_n$ may be both converge or diverge.

Since $\sum v_n$ is converge only if $k > 1$ and diverge if $k \leq 1, \sum u_n$ is converge only if $x < 1$ and diverge if $x > 1$ for all values of k.

Also if $x = 1$, then the series converges only if $k > 1$ and diverge if $k < 1$ for all values of k.

3) Discuss the convergence of the series $\frac{1}{1+x} + \frac{1}{1+2x^2} + \frac{1}{1+3x^3} + \cdots$

Solution:

Given $\sum u_n = \frac{1}{1+x} + \frac{1}{1+2x^2} + \frac{1}{1+3x^3} + \cdots$

Let $u_n = \frac{1}{1+nx^n}$. Then $u_{n+1} = \frac{1}{1+(n+1)x^{n+1}}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{1}{1+(n+1)x^{n+1}}.(1+nx^n) = \frac{\frac{1}{nx^n}+1}{\frac{1}{nx^n}+x\left(1+\frac{1}{n}\right)}$$

If $x = 1$, the series becomes $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n}$ which is divergent.

If $x < 1, 1 + x < 2 \Rightarrow \frac{1}{1+x} > \frac{1}{2}$

$1 + 2x^2 < 3 \Rightarrow \frac{1}{1+2x^2} > \frac{1}{3}$

$1 + 3x^3 < 3 \Rightarrow \frac{1}{1+3x^3} > \frac{1}{4}$ and so on

Hence $\sum u_n$ is convergent only if $x > 1$ and divergent if $x \leq 1$.

4) Examine the convergence of the series $\sum_{n=0}^{\infty} \left(\frac{n}{n+1}\right)^{1/2} x^n$

5) Settle the range of values of x for which the following series converge:

(a) $\sum \frac{x^n}{1+x^n}$

(b) $\frac{3}{8} + \frac{3.5}{8.10} + \frac{3.5.7}{8.10.12} + \cdots$

(c) $\left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 + \cdots$

(d) $\frac{2}{3} + \frac{2.3}{3.5} + \frac{2.3.4}{3.5.7} + \cdots$

Raabe's Test:

Theorem:1

If $\sum u_n$ and $\sum v_n$ are two series of positive terms and if $\frac{u_{n+1}}{u_n} < \frac{v_{n+1}}{v_n}$ for all values of n after a certain stage, show that $\sum u_n$ will converge if $\sum v_n$ converges.

Proof:

Since the omission of a finite number of terms form a series does not affect convergence. We can assume that the inequality holds for all positive integer values of n.Thus

$$\frac{u_2}{u_1} < \frac{v_2}{v_1}, \frac{u_3}{u_2} < \frac{v_3}{v_2}, \frac{u_4}{u_3} < \frac{v_4}{v_3}, \ldots$$

$$u_1 + u_2 + u_3 + \cdots = u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \cdots\right) < u_1\left(1 + \frac{v_2}{u_1} + \frac{v_3}{u_1} + \frac{u_4}{u_1} + \cdots\right)$$

$$< u_1\left(1 + \frac{v_2}{v_1} + \frac{v_3}{v_2} + \frac{u_4}{v_3} + \cdots\right)$$

$$< u_1\left(1+\frac{v_2}{v_1}+\frac{v_3}{v_1}+\frac{u_4}{v_1}+\cdots\right)$$
$$< \frac{u_1}{v_1}(v_1+v_2+v_3+\cdots)$$

Since $\frac{u_1}{v_1}$ is a constant, and $\sum v_n$ is convergent and it follows that $\sum u_n$ is convergent.

Theorem:2

If $\sum v_n$ diverges and if $\frac{u_{n+1}}{u_n} > \frac{v_{n+1}}{v_n}$ then $\sum u_n$ diverges.

Proof:

We have

$$\frac{u_2}{u_1} > \frac{v_2}{v_1}, \frac{u_3}{u_2} > \frac{v_3}{v_2}, \frac{u_4}{u_3} > \frac{v_4}{v_3}, \ldots$$

$$u_1+u_2+u_3+\cdots = u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\frac{u_4}{u_1}+\cdots\right) = u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}.\frac{u_2}{u_1}+\frac{u_4}{u_3}.\frac{u_3}{u_2}.\frac{u_2}{u_1}+\cdots\right)$$
$$> u_1\left(1+\frac{v_2}{v_1}+\frac{v_3}{v_2}.\frac{v_2}{v_1}+\frac{v_4}{v_3}.\frac{v_3}{v_2}.\frac{v_2}{v_1}+\cdots\right)$$
$$> u_1\left(1+\frac{v_2}{v_1}+\frac{v_3}{v_2}+\frac{u_4}{v_3}+\cdots\right)$$
$$> \frac{u_1}{v_1}(v_1+v_2+v_3+\cdots)$$

Since $\frac{u_1}{v_1}$ is a constant, and $\sum v_n$ is divergent and it follows that $\sum u_n$ is divergent.

Raabe's Test:

Let us compare the series $\sum u_n$ with the series $\sum \frac{1}{n^p}$.

Proof:

$\sum \frac{1}{n^p}$ is convergent when $p > 1$ and divergent if $p \leq 1$.

$\sum u_n$ is convergent,if $\frac{u_{n+1}}{u_n} < \frac{n^p}{(n+1)^p}$.

$$\text{i.e., if } \frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p},$$

$$\text{i.e.,} \quad \text{if } \frac{u_n}{u_{n+1}} > (1+\frac{1}{n})^p > 1+\frac{p}{n}+\frac{p(p-1)}{2n^2}+\cdots \text{(by binomial theorem)}$$

$$\text{i.e.,} \quad \text{if n}\left(\frac{u_n}{u_{n+1}}-1\right) > p+\frac{p(p-1)}{2n}+\cdots$$

$$\text{i.e.,} \quad \text{if } \lim_{n\to\infty} \text{n}\left(\frac{u_n}{u_{n+1}}-1\right) > p$$

But the auxiliary series is convergent if $p > 1$ which shows that $\sum u_n$ is convergent when $p > 1$.

And $\sum u_n$ is divergent if $\frac{u_n}{u_{n+1}} < \frac{n^p}{(n+1)^p}$

$$\text{i.e., if } \frac{u_n}{u_{n+1}} < (1+\frac{1}{n})^p,$$

$$\text{i.e.,} \quad \text{if } \frac{u_n}{u_{n+1}} < (1+\frac{1}{n})^p < 1+\frac{p}{n}+\frac{p(p-1)}{2n^2}+\cdots \text{(by binomial theorem)}$$

$$\text{i.e.,} \quad \text{if n}\left(\frac{u_n}{u_{n+1}}-1\right) < p+\frac{p(p-1)}{2n}+\cdots$$

i. e., if $\lim_{n\to\infty} \mathrm{n}\left(\frac{u_n}{u_{n+1}} - 1\right) < p$, which shows that $\sum u_n$ is divergent when $p < 1$.

This test can be enunciated as follows:

The series whose general terms u_n is convergent or divergent according as

$$\lim_{n\to\infty} \left\{\mathrm{n}\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} > 1 \text{ or } < 1.$$

This is known as Raabe's test.

Examples:

1) Prove that the series $1 + \frac{1}{2}.\frac{a}{b} + \frac{1.3.a(a+1)}{2.4.b(b+1)} + \frac{1.3.a(a+1)(a+2)}{2.4.b(b+1)(b+2)} + \cdots$ is convergent if $a > 0, b > 0$ and $b > a + \frac{1}{2}$.

Proof:

Let u_n denote the n^{th} terms of the series. Then for $n > 1$,

$$u_n = \frac{1.3.5 \ldots (2n-3)}{2.4.6 \ldots (2n-2)} \cdot \frac{a(a+1)(a+2) \ldots (a+n-2)}{b(b+1)(b+2) \ldots (b+n-2)}$$

Hence,

$$\frac{u_{n+1}}{u_n} = \frac{(2n-1)(n-1+a)}{2n(n-1+b)}$$

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{\left(1 - \frac{1}{2n}\right)\left(1 + \frac{a-1}{n}\right)}{\left(1 + \frac{b-1}{n}\right)}$$

Thus D'Alembert's ratio test fails.

Again,

$$\frac{u_n}{u_{n+1}} - 1 = \frac{\left((2b - 2a + 1) + \frac{a-1}{n}\right)}{n\left(2 - \frac{1}{n}\right)\left(1 + \frac{a-1}{n}\right)}$$

Hence,

$$\lim_{n\to\infty} \left\{\mathrm{n}\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = b - a + \frac{1}{2} > 1$$

Thus the series converges if $b - a + \frac{1}{2} > 1$.

2) Discuss the convergence of the series $1 + \frac{(1!)^2}{2!}x + \frac{(2!)^2}{4!}x^2 + \cdots + \frac{(n!)^2}{(2n)!}x^n + \cdots$

Solution:

Leaving the first term $u_n = \frac{(n!)^2}{(2n)!}x^n$

Hence,

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)}{2(2n+1)}x$$

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{x}{4}$$

Hence the series converges if $x < 4$ and diverges if $x > 4$.

If $x = 4$, then D'Alembert's ratio test fails.

Hence,

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}=\lim_{n\to\infty}\left\{n\left(\frac{2(2n+1)}{4(n+1)}-1\right)\right\}=\lim_{n\to\infty}=\frac{-n}{2(n+1)}=-\frac{1}{2}<1.$$

Thus, the series diverges if $x=4$.

Corollary:1

The series whose general terms u_n is convergent or divergent according as

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}>1 \text{ or } <1.$$

Corollary:2

The series whose general terms u_n is convergent or divergent according as

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\,log\,n\right\}>1 \text{ or } <1.$$

<u>Do it:</u>

1) **Test for convergency and divergency of the series** $1+\frac{2x}{2!}x+\frac{(3x)^2}{3!}x^2+\frac{(4x)^3}{4!}x^3\ldots+\frac{(n!)^2}{(2n)!}x^{n-1}+\cdots$
2) **Examine the convergence of** $\left(\frac{1}{2}\right)^2+\left(\frac{1.3}{2.4}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2+\cdots$

Geometric series:

Consider the geometric series $1+x+x^2+\cdots+x^n$

Let $S_n=1+x+x^2+\cdots+x^n=\frac{1-x^n}{1-x}$

Case(i)

$0\le x<1.$ Then $(\mathrm{x^n})\to 0\Rightarrow(S_n)\to\frac{1}{1-x}$.

$\Rightarrow$ the given series converges to the sum $\frac{1}{1-x}$

Case(ii)

$x>1.$ Then $(S_n)\to\frac{x^n-1}{x-1}$. Also $(\mathrm{x^n})\to\infty$, when $\mathrm{x}>1$

$\Rightarrow$ the given series diverges to the sum ∞.

Case(iii)

$x=1.$ Then the series becomes $1+1+\cdots+1$(n times) $\Rightarrow\ (S_n)\to(n)$ and hence $(\mathrm{S_n})\to\infty$, when $\mathrm{x}=1$

$\Rightarrow$ the given series diverges to the sum ∞.

Case(iv)

$x=-1.$ Then the series becomes $1-1+1-1\ldots$

$\Rightarrow\ (S_n)\to\begin{cases}0, & \text{if } n \text{ is even}\\ 1, & \text{if } n \text{ is odd}\end{cases}$ and hence $(\mathrm{S_n})$ oscillates finitely.

Hence the given series oscillates finitely.

Case(v)

$x<-1.$ Then $(\mathrm{x^n})$ oscillates infinitely and hence $(\mathrm{S_n})$ oscillates infinitely.

Hence the given series oscillates infinitely.

Thus the geometric series converges if $0\le x<1$, diverges if $x>1$ and oscillates if $x\le -1$.

Theorem:

Discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{1}{n^k}$. (i.e., $\frac{1}{1^k}+\frac{1}{2^k}+\frac{1}{3^k}+\cdots$ is convergent when k is greater than unity and divergent when k is equal to or less than unity)

Proof:

Given $\sum_{n=1}^{\infty}\frac{1}{n^k}$

Case(i): when $k > 1$

The first term of the given series is $\frac{1}{1^k}=1$.

Consider the following,

$$\frac{1}{2^k}+\frac{1}{3^k}<\frac{1}{2^k}+\frac{1}{3^k}=\frac{2}{2^k}=\frac{1}{2^{k-1}}.$$

And

$$\frac{1}{4^k}+\frac{1}{5^k}+\frac{1}{6^k}+\frac{1}{7^k}<\frac{1}{4^k}+\frac{1}{4^k}+\frac{1}{4^k}+\frac{1}{4^k}=\frac{4}{4^k}=\frac{1}{4^{k-1}}=\frac{1}{(2^2)^{k-1}}=\left(\frac{1}{2^{k-1}}\right)^2.$$

Similarly,

$$\frac{1}{8^k}+\frac{1}{9^k}+\frac{1}{10^k}+\frac{1}{11^k}+\frac{1}{12^k}+\frac{1}{13^k}+\frac{1}{14^k}+\frac{1}{15^k}+\frac{1}{16^k}$$
$$<\frac{1}{8^k}+\frac{1}{8^k}+\frac{1}{8^k}+\frac{1}{8^k}+\frac{1}{8^k}+\frac{1}{8^k}+\frac{1}{8^k}+\frac{1}{8^k}+\frac{1}{8^k}=\frac{8}{8^k}=\frac{1}{8^{k-1}}$$
$$=\frac{1}{2^{3(k-1)}}=\left(\frac{1}{2^{k-1}}\right)^3$$

Hence the whole term in the series, i.e.,

$$\frac{1}{1^k}+\left(\frac{1}{2^k}+\frac{1}{3^k}\right)+\left(\frac{1}{4^k}+\frac{1}{5^k}+\frac{1}{6^k}+\frac{1}{7^k}\right)+\cdots<\frac{1}{1}+\frac{1}{2^{k-1}}+\left(\frac{1}{2^{k-1}}\right)^2+\left(\frac{1}{2^{k-1}}\right)^3+\cdots \quad ---(1)$$

The right-hand series is a geometric series with common ratio $\frac{1}{2^{k-1}}$ which is less than 1 as $k \geq 1$. We know that any geometric series is convergent if the common ratio is less than 1. Hence the series in the right-hand series of (1) is convergent.

Hence by comparison test, $\sum_{n=1}^{\infty}\frac{1}{n^k}$ is convergent $k > 1$.

Case(ii): when $k = 1$

Then the given series is equal to $\sum_{n=1}^{\infty}\frac{1}{n^1}=1+\frac{1}{2}+\frac{1}{3}+\cdots$.

Consider the following,

$$1=\frac{1}{2}+\frac{1}{2};$$
$$\frac{1}{2}=\frac{1}{2};$$
$$\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2};$$
$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{4}{8}=\frac{1}{2};$$

$$\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\cdots+\frac{1}{16}>\frac{8}{16}=\frac{1}{2};$$

$$\cdots$$

$$\cdots$$

We can group the series as follows:

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots>1+n\left(\frac{1}{2}\right)=1+\left(\frac{n}{2}\right).$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n}>1+\frac{n}{2},\forall\, n.$$

i.e., the series greater that $\frac{n}{2}+1$, which increase indefinitely with n.

Hence the whole term in the series, i.e.,

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}\text{ is divergent.}$$

Case(iii): when $k\leq 1$

In this case, $\frac{1}{n^k}>\frac{1}{n}$. Thus every term of the series $\sum_{n=1}^{\infty}\frac{1}{n^k}$(after the first term) is greater than the corresponding term of the known divergent series $\sum_{n=1}^{\infty}\frac{1}{n}$. Thus $\sum_{n=1}^{\infty}\frac{1}{n^k}$ is divergent when $k<1$.

Unit III

Cauchy's condensation test:

If (n) is positive, for all positive integral values of n and continually diminishes as n increases and if "a" be any positive integer, then two infinite series $(1) + f(2) + f(3) + \cdots + f(n) + \cdots$ and $a\, f(a) + a^2 f(a^2) + \cdots . + a^n(a^n) + \cdots$ are both convergent or both divergent.

Proof:

$$\text{Let } \sum (n) = (f(1) + f(2) + f(3) + \cdots + f(a)) + (f(a+1) + f(a+2) + (a+3) + \cdots + f(a^2)) + (f(a^2+1) + f(a^2+2) + f(a^2+3) + \cdots + f(a^3)) + \cdots + ((a^{n-1}+1) + f(a^{n-1}+2) + f(a^{n-1}+3) + \cdots + f(a^n))$$

Let v_n denote the terms of the n^{th} group

$$(a^{n-1}+1) + f(a^{n-1}+2) + f(a^{n-1}+3) + \cdots + f(a^n)$$

The number of terms in this group is $a^n - a^{n-1}$.

Since (n) is a decreasing function,

$$(a^n - a^{n-1})(a^n) \le v_n \le (a^n - a^{n-1}) f(a^{n-1})$$

$$\Rightarrow \frac{1}{a}(a-1)^n f(a^n) \le v_n \le (a-1) a^{n-1} f(a^{n-1})$$

Now, if $\sum a^n(a^n)$ is convergent, then so is $\sum v_n$(taking the right-hand half of last inequality)

$$\sum (n) \text{ is convergent.}$$

if $\sum a^n f(a^n)$ is divergent, then so is $\sum v_n$(taking the left-hand half of first inequality)

$$\sum (n) \text{ is divergent.}$$

$$\Rightarrow \sum (n)\ \&\ \sum a^n f(a^n) \text{ are both convergent or both divergent.}$$

Problems:

1) Show that the series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is divergent.

Solution:

Let $(n) = \frac{1}{n}$.

$$\text{Then, } \sum (n) = \sum \frac{1}{n}\ \&\ \sum a^n(a^n) = \sum 2^n f(2^n), \text{by taking a} = 2.$$

$$\Rightarrow \sum 2^n \frac{1}{2^n} = \sum 1 = 1 + 1 + \cdots$$

$$\Rightarrow \sum \frac{1}{n} \text{ and } \sum 1 \text{ behave alike.}$$

But $1 + 1 + \cdots$ is divergent.

$$\Rightarrow \sum \frac{1}{n} \text{is divergent.}$$

2) Examine the convergence of the series

a) $\sum \frac{1}{n^k}$

Solution:

Let $(n) = \frac{1}{n^k}$.

Then, $\sum (n)$ & $\sum a^n f(a^n)$ behave alike.

$$\Rightarrow \sum a^n(a^n) = \sum 2^n \frac{1}{(2^n)^k} = \sum \frac{1}{2^{(k-1)}}, \text{by taking a} = 2.$$

which is a geometric series and it is convergent or divergent according to $k > 1$ or $k \leq 1$.

$$\Rightarrow \sum \frac{1}{n^k} \text{ is convergent or divergent according to } k > 1 \text{ or } k \leq 1.$$

b) $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \cdots + \frac{1}{(2n+1)(2n+2)} + \cdots$

Solution:

Let $(n) = \frac{1}{(2n+1)(2n+2)}$.

Then, $\sum (n)$ & $\sum a^n f(a^n)$ behave alike.

$$\Rightarrow \sum a^n(a^n) = \sum 2^n \frac{1}{(2n+1)(2n+2)}, \text{by taking a} = 2.$$

$$\Rightarrow \sum a^n(a^n) = \sum \frac{1}{2^n\left(2+\frac{1}{2^n}\right)\left(2+\frac{2}{2^n}\right)}$$

$$\text{Let } U_n = \frac{1}{2^n\left(2+\frac{1}{2^n}\right)\left(2+\frac{2}{2^n}\right)} \Rightarrow \lim_{n\to\infty} U_n = 0.$$

$$\Rightarrow \sum \frac{1}{(2n+1)(2n+2)} \text{ is convergent.}$$

c) $1 + \frac{1}{3} + \frac{1}{5} + \cdots$

Solution:

Let $(n) = \frac{1}{(2n-1)}$.

Then, $\sum (n)$ & $\sum a^n f(a^n)$ behave alike.

$$\Rightarrow \sum a^n(a^n) = \sum 2^n \frac{1}{(2.2^n - 1)}, \text{by taking a} = 2.$$

$$\Rightarrow \sum a^n f(a^n) = \sum \frac{2^n}{2^n\left(2-\frac{1}{2^n}\right)} = \sum \frac{1}{\left(2-\frac{1}{2^n}\right)}$$

$$\text{Let } U_n = \frac{1}{\left(2-\frac{1}{2^n}\right)} \Rightarrow \lim_{n\to\infty} U_n = \frac{1}{2} \neq 0.$$

$$\Rightarrow \sum \frac{1}{(2n-1)} \text{ is divergent.}$$

d) $\frac{1.2}{3.4.5}+\frac{2.3}{4.5.6}+\cdots$

Solution:

$$\text{Let } (n) = \frac{(n+1)}{(n+2)(n+3)(n+4)}.$$

$$\text{Then, } \sum (n) \quad \& \quad \sum a^n f(a^n) \text{ behave alike.}$$

$$\Rightarrow \sum a^n f(a^n) = \sum \frac{(1+\frac{1}{2^n})}{(1+\frac{2}{2^n})(1+\frac{3}{2^n})(1+\frac{4}{2^n})}, \text{by taking a} = 2.$$

$$\text{Let } U_n = \frac{(1+\frac{1}{2^n})}{(1+\frac{2}{2^n})(1+\frac{3}{2^n})(1+\frac{4}{2^n})} \Rightarrow \lim_{n\to\infty} U_n = 1 \neq 0.$$

$$\Rightarrow \sum \frac{(n+1)}{(n+2)(n+3)(n+4)} \text{is divergent.}$$

e) $\sum \frac{1}{n(\log n)^p}$

Solution:

$$\text{Let } (n) = \frac{1}{n(\log n)^p}.$$

$$\text{Then, } \sum (n) \quad \& \quad \sum a^n f(a^n) \text{ behave alike.}$$

$$\Rightarrow \sum a^n (a^n) = \sum \mathrm{a^n} \frac{1}{\mathrm{a^n}(\log n)^p}, \text{by taking n} = \mathrm{a^n}.$$

$$\Rightarrow \sum a^n (a^n) = \sum \frac{1}{(\log a^n)^p}$$

$$\Rightarrow \sum a^n (a^n) = \sum \frac{1}{(n \log a)^p}$$

Then, $\sum \frac{1}{(\log n)^p}$ & $\sum \frac{1}{(n\ \log a)^p}$ both converge or diverge together and the second series becomes,

$$\sum \frac{1}{(n\ \log a)} = \sum \frac{1}{n^p(\log a)^p} = \frac{1}{\log a^p} \sum \frac{1}{n^p}$$

$$\Rightarrow \sum \frac{1}{(\log n)^p} \text{converge or diverge according to p,}$$

i. e., if $p \leq 1$, then $\sum \frac{1}{n^p}$ is divergent and if $p > 1$, then $\sum \frac{1}{n^p}$ is divergent.

$f) \frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots$

Solution:

Let $(n) = \frac{(n+1)}{n^p}$.

$$\text{Then, } \sum (n) \quad \& \quad \sum a^n f(a^n) \text{ behave alike.}$$

$$\Rightarrow \sum a^n f(a^n) = \sum \frac{2(2^n+1)}{2^{np}}, \text{ by taking a} = 2.$$

$$\Rightarrow \sum a^n f(a^n) = \sum \frac{2^n . 2^n \left(1+\frac{1}{2^n}\right)}{2^{np}} = \sum \frac{\left(1+\frac{1}{2^n}\right)}{2^{np}. 2^{-2n}} = \sum \frac{\left(1+\frac{1}{2^n}\right)}{2^{(p-2)}}$$

$$\Rightarrow \sum \frac{(n+1)}{n^p} \text{ is convergent if p} > 1 \text{ and is divergent, if p} \leq 1.$$

Cauchy's root test:

If $\sum_{n=1}^{\infty} U_n$ is a series of terms, Prove that the series is convergent or divergent according as $\lim_{n\to\infty} U_n^{\frac{1}{n}} < 1$ or > 1.

Proof:

Case(i) $\lim_{n\to\infty} U_n^{\frac{1}{n}} = l, l < 1$.

Hence we can choose ϵ, positive and sufficiently small, so that $l + \epsilon < 1$.

Since $\lim_{n\to\infty} U_n^{\frac{1}{n}} = l$, we can find a natural number "m" so large so that $U_n^{\frac{1}{n}}$ differ from *** by less than ϵ, so long $n \geq m$.

$$\therefore U_n^{\frac{1}{n}} < l + \epsilon \Rightarrow U_n < (l+\epsilon)^n.$$

Hence from and after the m^{th} term of the series $\sum_{n=1}^{\infty} U_n$ are less than those of the geometric series $\sum (l+\epsilon)^n$ which is convergent since $l + \epsilon < 1$.

$$\Rightarrow \sum_{n=1}^{\infty} U_n \text{ is convergent.}$$

Case(ii) $\lim_{n\to\infty} U_n^{\frac{1}{n}} = l, l > 1$.

Hence we can choose ϵ, positive and sufficiently small, so that $l - \epsilon > 1$.

Since $\lim_{n\to\infty} U_n^{\frac{1}{n}} = l$, we can find a natural number "m" so large so that $U_n^{\frac{1}{n}}$ differ from *** by less than ϵ, so long $n \geq m$.

$$\therefore U_n^{\frac{1}{n}} < l - \epsilon \Rightarrow U_n > (l-\epsilon)^n.$$

Hence from and after the m^{th} term of the series $\sum_{n=1}^{\infty} U_n$ are greater than those of the geometric series $\sum (l+\epsilon)^n$ which is divergent since $l - \epsilon > 1$.

$$\Rightarrow \sum_{n=1}^{\infty} U_n \text{ is divergent.}$$

$$\text{Thus } \lim_{n\to\infty} U_n^{\frac{1}{n}} = \begin{cases} \sum u_n \text{ is convergent}, & \text{if } l < 1. \\ \sum u_n \text{ is divergent}, & \text{if } l > 1. \\ \text{the test fails}, & \text{if } l = 1. \end{cases}$$

Problems:

1) Test the convergence of the following series:

a) $\boldsymbol{a + b + a^2 + b^2 + a^3 + b^3 + \cdots}$

Solution:

$$\text{Let } U_n = \begin{cases} a^{\frac{n+1}{2}}, & \text{when n is odd} \\ b^{\frac{n}{2}} & \text{when n is even} \end{cases} \quad \text{or}$$

$$\Rightarrow U_n^{\frac{1}{n}} = \begin{cases} \left(a^{\frac{n+1}{2}}\right)^{1/n}, & \text{when n is odd} \\ \left(b^{\frac{n}{2}}\right)^{\frac{1}{n}}, & \text{when n is even} \end{cases} \quad \text{or}$$

$$\Rightarrow U_n^{\frac{1}{n}} = \begin{cases} a^{\frac{1}{2}+\frac{1}{2n}}, & \text{when n is odd} \\ b^{\frac{1}{2}} & \text{when n is even} \end{cases} \quad \text{or}$$

$$\Rightarrow \lim_{n\to\infty} U_n^{\frac{1}{n}} = \begin{cases} a^{\frac{1}{2}+\frac{1}{2n}}, & \text{when n is odd} \\ b^{\frac{1}{2}} & \text{when n is even} \end{cases} \quad \text{or}$$

$$\Rightarrow \lim_{n\to\infty} U_n^{\frac{1}{n}} = \begin{cases} a^{\frac{1}{2}}, & \text{when n is odd} \\ b^{\frac{1}{2}} & \text{when n is even} \end{cases} \quad \text{or}$$

Thus the series converges if $0 < a < 1$, $0 < b < 1$ and diverges if $a \geq 1$ or $b \geq 1$.

b) $\boldsymbol{\sum \left(\frac{(n+1)(n+2)\ldots(n+n)}{n^n}\right)}$

Solution:

$$\text{Let } U_n = \frac{(n+1)(n+2)\ldots(n+n)}{n^n}$$

$$\Rightarrow U_n^{\frac{1}{n}} = \left(\frac{(n+1)(n+2)\ldots(n+n)}{n^n}\right)^{\frac{1}{n}}$$

$$\Rightarrow U_n^{\frac{1}{n}} = \left(n^n \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)}{n^n}\right)^{\frac{1}{n}}$$

$$\Rightarrow \lim_{n\to\infty} U_n^{\frac{1}{n}} = \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right)^{\frac{1}{n}}$$

Let this limit be "l".

$$\Rightarrow \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right)^{\frac{1}{n}} = l$$

Taking log on both sides,

$$\Rightarrow \log\left\{\lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right)^{\frac{1}{n}}\right\} = \log l$$

$$\Rightarrow \log l = \lim_{n\to\infty} \sum_{r=1}^{n} \left(1+\frac{r}{n}\right)$$

$$\Rightarrow \log l = \int_0^1 \log(1+x)\,dx$$

$$u = \log(1+x) \quad \int dv = \int dx$$

$$du = \frac{1}{1+x}dx \qquad v = x$$

By the formula, $\int u\,dv = uv - \int v\,du$, we have,

$$\log l = (x\log(1+x))_0^1 - \int x.\left(\frac{1}{1+x}\right)dx$$

$$\Rightarrow \log l = \log 2 - \int_0^1 \frac{x}{1+x}dx$$

$$\Rightarrow \log l = \log 2 - \int_0^1 \frac{x+1-1}{1+x}dx$$

$$\Rightarrow \log l = \log 2 - \int_0^1 \frac{x+1}{1+x}dx - \int_0^1 \frac{x}{1+x}dx$$

$$\Rightarrow \log l = \log 2 - (x - \log(1+x))_0^1 + c$$

$$\Rightarrow \log l = \log 2 - [(1-\log 2) - (0 - \log 1)]$$

$$\Rightarrow \log l = \log 2 - [(1-\log 2)]$$

$$\Rightarrow \log l = 2\log 2 - 1 = \log_e 4 - \log_e e$$

$$\Rightarrow \log l = \log_e\left(\frac{4}{e}\right) \Rightarrow l = \frac{4}{e}.$$

Hence $\sum U_n$ diverges.

2) Examine the convergence of the following series:

a) $\sum \frac{x^n}{n^n}$

Solution:

$$\text{Let } U_n = \frac{x^n}{n^n}$$

$$\Rightarrow U_n^{\frac{1}{n}} = \left(\frac{x^n}{n^n}\right)^{\frac{1}{n}} = \frac{x}{n}$$

$$\Rightarrow \lim_{n\to\infty} U_n^{\frac{1}{n}} = 0 < 1$$

Hence $\sum U_n$ converges.

b) $\sum \frac{1}{(\log n)}$

Solution:

$$\text{Let } U_n = \frac{1}{(\log n)}$$

$$\Rightarrow U_n^{\frac{1}{n}} = \left(\frac{1}{(\log n)}\right)^{\frac{1}{n}} = \frac{1}{\infty} = 0 < 1$$

Hence $\sum U_n$ converges.

c) $\sum(\sqrt{n}-1)^{n}$

Solution:

$$\text{Let } U_n = (\sqrt{n}-1)^{n}$$
$$\Rightarrow U_n^{\frac{1}{n}} = \sqrt{n}-1$$
$$\Rightarrow \lim_{n\to\infty} U_n^{\frac{1}{n}} = \sqrt{\infty}-1 = \infty - 1 = \infty$$
$$\text{Hence } \sum U_n \text{ diverges.}$$

d) $\sum \frac{1}{(1+\frac{1}{n})^{n^2}}$

Solution:

$$\text{Let } U_n = \frac{1}{(1+\frac{1}{n})^{n^2}}$$
$$\Rightarrow U_n^{\frac{1}{n}} = \frac{1}{(1+\frac{1}{n})^{n}}$$
$$\Rightarrow \lim_{n\to\infty} U_n^{\frac{1}{n}} = \frac{1}{e} < 1$$
$$\text{Hence } \sum U_n \text{ converges.}$$

3) **Investigate the behaviour of the series whose general term is $\frac{n!}{n^n}$.**

Solution:

$$\text{Let } U_n = \frac{n!}{n^n}$$
$$\Rightarrow U_n^{\frac{1}{n}} = \left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$$
$$\Rightarrow \lim_{n\to\infty} U_n^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1}{n}\cdot\frac{2}{n}\cdots\frac{n}{n}\right)^{\frac{1}{n}}$$
$$\text{Now } \log \lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{n}\sum_{n=1}^{n}\log\frac{n}{n} = \int_0^1 \log x\, dx$$
$$u = \log x \qquad \int dv = \int dx$$
$$du = \frac{1}{x}dx \qquad v = x$$

By the formula, $\int u\, dv = uv - \int v\, du$, we have,

$$\log \lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{\frac{1}{n}} = (x\log x)_0^1 - \int x.\left(\frac{1}{x}\right) dx$$
$$\text{Hence } \sum U_n \text{ diverges.}$$

4) Show that the series $\sum \left(\frac{((n+1)r)^n}{n^{n+1}}\right)$ is convergent if $r < 1$ and divergent if $r \geq 1$.

Solution:

$$\text{Let } U_n = \frac{((n+1))^n}{n^{n+1}}$$
$$\Rightarrow U_n^{\frac{1}{n}} = \frac{(n+1)r}{n^{n+1}}$$
$$\Rightarrow \lim_{n\to\infty} U_n^{\frac{1}{n}} = r$$

If $r < 1$, the series is convergent.

If $r > 1$, the series is divergent.

If $r < 1$, the test fails.

$$\text{Now, } U_n = \frac{((n+1))^n}{n^{n+1}}$$
$$\Rightarrow U_n = \frac{(1+\frac{1}{n})^n}{n}$$
$$\text{Take } V_n = \frac{1}{n}$$
$$\Rightarrow \frac{U_n}{V_n} = \frac{(1+\frac{1}{n})^n}{n} \text{ x n} = (1+\frac{1}{n})^n$$
$$\Rightarrow \lim_{n\to\infty} \frac{U_n}{V_n} = e.$$

Hence $\sum U_n$ & $\sum V_n$ both converge or diverge together. But $\sum V_n$ is divergent if $r > 1$ and convergent if $r < 1$ implies $\sum U_n$ is divergent if $r > 1$ and convergent if $r < 1$.

Alternating series:

I. Absolutely convergent series:

The series $\sum U_n$ containing positive and negative terms, is said to be absolutely convergent, if the series formed by the numerical values of the terms of $\sum U_n$. i.e., $\sum |U_n|$ is convergent.

Example:

The series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ is absolutely convergent, since the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ is convergent.

Problems:

1) Test the convergence of the series:

(a) $\sum \frac{2^n n!}{n^n}$

Solution:

$$Let\ U_n = \frac{2^n n!}{n^n}$$
$$\Rightarrow U_n^{\frac{1}{n}} = \left(\frac{2^n n!}{n^n}\right)^{\frac{1}{n}}$$

$$\Rightarrow U_n^{\frac{1}{n}} = 2\left(\frac{1.2.3 \dots n}{n.n \dots n}\right)^{\frac{1}{n}}$$

$$\Rightarrow \lim_{n\to\infty} U_n^{\frac{1}{n}} = 2 \lim_{n\to\infty} \left(\frac{1.2.3 \dots n}{n.n \dots n}\right)^{\frac{1}{n}}$$

$$\Rightarrow \lim_{n\to\infty} U_n^{\frac{1}{n}} = 2 \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \log\frac{n}{n} = 2\int_0^1 \log x \, dx$$

$$u = \log x \qquad \int dv = \int dx$$

$$du = \frac{1}{x}dx \qquad v = x$$

By the formula, $\int u\,dv = uv - \int v\,du$, we have,

$$2\int_0^1 \log x\,dx = (x\log x)_0^1 - \int x.\left(\frac{1}{x}\right)dx$$

$$\Rightarrow 2\int_0^1 \log x\,dx = 2[0-(x)_0^1] = -2\log_e e = 2\log_e e^{-1} = 2e^{-1} = \frac{2}{e} = \frac{2}{2.71}.$$

$$\Rightarrow 2\int_0^1 \log x\,dx = 0.71 < 1$$

Hence $\sum U_n$ converges.

(b) $\sum \frac{2^n n!}{n^n}$ **(Try this)**

II. Conditionally convergent series:

The series $\sum U_n$ containing positive and negative terms, is said to be conditionally convergent or semi-convergent, if the series $\sum U_n$ is convergent and $\sum |U_n|$ is convergent.

Example:

The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is conditionally convergent, since $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ is divergent.

Theorem:1

An absolutely convergent series is convergent.

Proof:

Let $\sum U_n$ be the given series. Then by hypothesis, $\sum |U_n|$ is convergent.

$$Now, U_n + |U_n| = \begin{cases} 2U_n, & \text{if } U_n \text{ is positive} \\ 0, & \text{if } U_n \text{ is negative} \end{cases}$$

∴ Every term of the series $\sum(U_n + |U_n|)$ is positive and is less than or equal to the corresponding terms of the convergent series $2\sum|U_n|$.

Hence $\sum(U_n + |U_n|)$ is convergent.

Since $\sum |U_n|$ is convergent, the series formed by the difference of the corresponding terms of both the series is convergent and hence $\sum U_n$ is convergent.

Note:

(1) When we say that U_n is absolutely convergent, we assert the convergence of another series $\sum |U_n|$ and not that of $\sum U_n$ alone.

(2) If after a stage $\left|\frac{U_{n+1}}{U_n}\right| < k$ or $\left|\frac{U_{n+1}}{U_n}\right|^{\frac{1}{n}} < k$, where k is fixed positive number less than infinity, then $\sum |U_n|$ is convergent. Therefore, if one of the above conditions is satisfied, then $\sum U_n$ is absolutely convergent.

(3) If two series are absolutely convergent, then they can be multiplied and the resulting series is also an absolutely convergent series.

Theorem:2

If the term often absolutely convergent series are rearranged, then the series remains convergent and its sum is unaltered.

Theorem:3

In a conditionally convergent series, a rearrangement of the term alter its sum.

Series whose terms are alternately positive and negative.

Theorem:4

If $\boldsymbol{U_1 - U_2 + U_3 - U_4 + \cdots}$ is a series of terms alternatively positive and negative and if $\boldsymbol{U_n > U_{n+1}}$ for all values of n, and if $\boldsymbol{\lim_{n\to\infty} U_n = 0}$, then the series is convergent.

Proof:

Let S_{2n} denote the sum to $2n$ terms of the series. Then

$$(U_1 - U_2) + (U_3 - U_4) + \cdots + (U_{2n-1} - U_{2n}).$$

Since each bracket is positive, S_{2n} steadily increases as "n" increases.

$$\text{i.e.}, S_2 < S_4 < S_6 \ldots$$

Without altering the given order of the terms, the sum S_{2n} may be written in term

$$S_{2n} = U_1 - (U_2 + U_3) - (U_4 - U_5) + \cdots + (U_{2n-2} - U_{2n-1}) - U_{2n}.$$

Since each bracket is positive, S_{2n} less than U_1, $\lim_{n\to\infty} S_{2n}$ exist and equal to "l" where $l < u_1$.

But $S_{2n+1} = S_{2n} + U_{2n+1}$ and $\lim_{n\to\infty} S_{2n+1} = l + 0$ implies $\lim_{n\to\infty} S_{2n+1} = l$.

Hence the series is convergent.

Problems:

1) Prove that the series $\boldsymbol{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots}$ is convergent

Solution:

In the given series,

(i) The terms are alternatively positive and negative.

(ii) $1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \cdots$i.e., the terms steadily decrease in numerical value.

(iii) $\lim_{n\to\infty} U_n = 0$.

Hence the three conditions for convergence are all satisfied and hence the series converges.

2) Discuss the convergence of the series $\sum(-1)^{n-1}\left(\frac{1}{n^p}\right)$ when $0 < p \le 1$.

Solution:

Given: $\sum(-1)^{n-1}\left(\frac{1}{n^p}\right)$ when $0 < p \le 1$.

If $p = 1$, the series becomes, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ which is convergent.

If $p = 1$, the series $\sum(-1)^{n-1}\left(\frac{1}{n^p}\right)$ becomes $\sum(-1)^{n-1} U_n$ where $U_n = \frac{1}{n^p}$.

$$\text{Now } U_n = \frac{1}{n^p} \Rightarrow U_{n+1} = \frac{1}{(n+1)}$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{1}{(n+1)^p} \times n^p$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{n^p}{n^p\left(1+\frac{1}{n}\right)^p} = \frac{1}{\left(1+\frac{1}{n}\right)^p}$$

$$\text{Since } p > 0 \ \&\ \left(1+\frac{1}{n}\right) > 1, \frac{U_{n+1}}{U_n} = \frac{1}{\left(1+\frac{1}{n}\right)^p} < 1.$$

$$\text{Thus } U_{n+1} < U_n$$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{n^p}, p > 0.$$

Hence the series converges.

3) Examine the convergence of the series $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \cdots, 0 < x < 1$.

Solution:

The terms of the given series are alternatively positive and negative.

$$\text{Now } U_n = \frac{x^n}{1+x^n} \Rightarrow U_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}$$

$$\Rightarrow U_n - U_{n+1} = \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}}$$

$$\Rightarrow U_n - U_{n+1} = \frac{x(1+x^{n+1}) - x^{n+1}(1+x^n)}{(1+x^n)(1+x^{n+1})}$$

$$\Rightarrow U_n - U_{n+1} = \frac{x(1-x)}{(1+x^n)(1+x^{n+1})}$$

Since x is positive and less than 1, $U_n - U_{n+1}$ = a positive quantity and hence $U_n = U_{n+1}$ + a positive quantity $\Rightarrow U_n > U_{n+1}$.

$$\text{Now } U_n = \frac{x^n}{1+x^n} \Rightarrow U_n = \frac{1}{x^n\left(1+\frac{1}{x^n}\right)} \Rightarrow \lim_{n\to\infty} U_n = 0, 0 < x < 1 \ \&\ x^n \to \infty \text{ as } n \to \infty$$

Hence the series converges.

4) Examine the convergence of $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \cdots$

Solution:

In the given series,

(i) The terms are alternatively positive and negative.

(ii) $1 > \frac{1}{5} > \frac{1}{9} > \frac{1}{13} > \cdots$ i.e., the terms steadily decrease in numerical value.

(iii) $\lim_{n\to\infty} U_n = 0.$

Hence the three conditions for convergence are all satisfied and hence the series converges.

5) Discuss the convergence of (a) $\sum(-1)^{n-1}\left(\frac{n}{n+1}\right)$.

Solution:

The terms of the given series are alternatively positive and negative.

$$\text{Now } U_n = \frac{n}{n+1} \Rightarrow U_{n+1} = \frac{n+1}{n+2}$$

$$\Rightarrow U_n - U_{n+1} = \frac{n}{n+1} - \frac{n+1}{n+2}$$

$$\Rightarrow U_n - U_{n+1} = \frac{(n+2) - (n+1)^2}{(n+1)(n+2)}$$

$$\Rightarrow U_n - U_{n+1} = \frac{n^2 + 2n - n^2 - 2n - 1}{(n+1)(n+2)} = -\frac{1}{(n+1)(n+2)}$$

This series is not steadily decreasing and $\lim_{n\to\infty} U_n = 1$.

Hence the series is not convergent.

Note:

It is important to note that if anyone of the conditions for convergent is removed, then the series need not be convergent.

Examples:

1) $1\frac{1}{2} - 1\frac{1}{4} + 1\frac{1}{8} - 1\frac{1}{16} + \cdots$. Hence the terms are alternatively positive and negative and steadily decreasing in numerical value.

$$\text{Let } U_n = 1 + \frac{1}{2^n} \Rightarrow \lim_{n\to\infty} U_n = 1 \neq 0 \Rightarrow \text{the series converges.}$$

2) $S_{2n} = \frac{1}{3}\left[1 - \left(\frac{1}{2}\right)^{2n}\right]$ & $S_{2n+1} = 1 + \frac{1}{3}\left[1 + \left(\frac{1}{2}\right)^{2n+1}\right]$

$$\therefore \text{when } n \to \infty, S_{2n} \to \frac{1}{3} \text{ and } S_{2n+1} \to 1.\frac{1}{3}$$

$$\Rightarrow \text{the series oscillates finitely.}$$

3) $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{8} + \frac{1}{4} - \frac{1}{16} + \cdots$. Hence the terms are alternatively positive and negative and $\lim_{n\to\infty} U_n = 0$. But the terms do not steadily decrease in numerical value. Thus the series is not convergent.

4) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is divergent and since $\frac{2}{2} + \frac{2}{4} + \frac{2}{8} + \cdots$ is convergent, the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is divergent. Here the terms steadily decreasing in numerical value and $\lim_{n\to\infty} U_n = 0$. But the terms are alternately positive and negative.

5) $\frac{1}{1.2}-\frac{1}{3.4}+\frac{1}{5.6}-\frac{1}{7.8}+\cdots$ is absolutely convergent.

The terms of the given series are alternatively positive and negative.

$$\text{Now } U_n=\frac{1}{2(2n-1)} \Rightarrow \ \mathrm{U}_{n+1}=\frac{1}{(2n+1)(2n+2)}$$

$$\Rightarrow \ \mathrm{U}_n-U_{n+1}=\frac{1}{2(2n-1)}-\frac{1}{(2n+1)(2n+2)}$$

$$\Rightarrow \ \mathrm{U}_n-U_{n+1}=\frac{(2n+1)(2n+2)-2(2n-1)}{2(2n-1)(2n+1)(2n+2)}$$

$$\Rightarrow \mathrm{U}_n-U_{n+1}=\frac{8n+2}{2(2n-1)(2n+1)(2n+2)}$$

$\Rightarrow \mathrm{U}_n-U_{n+1}=$ a positive quantity $\Rightarrow$ $\mathrm{U}_n>U_{n+1}+$ a positive quantity $\Rightarrow$ $\mathrm{U}_n>U_{n+1}$

Here the terms steadily decreasing in numerical value and $\lim_{n\to\infty} U_n=0$.

Thus the series is absolutely convergent.

Do it:

1) Test the convergence of the following series:

a) $\sum(-1)^n\ \left(\frac{n+1}{n}\right)$

b) $\sum(-1)^n\left(\frac{1}{n+8}\right)$

c) $1-\frac{1}{2}\sqrt{2}+\frac{1}{3}\sqrt{3}-\frac{1}{4}\sqrt{4}+\cdots$

d) $\sum(-1)^{n-1}\left(\frac{1}{x+n}\right)$

e) $\sum(-1)^{n-1}\left(\frac{1}{\sqrt{n-1}}\right), n>1$

f) $\sum(-1)^n\left(\frac{x^n}{n(n-1)}\right), n>1$

2) Show that $\frac{1}{1.2}-\frac{1}{3.4}+\frac{1}{5.6}-\frac{1}{7.8}+\cdots$ **is absolutely convergent.**

3) Test the absolute convergence of the series $\frac{1}{2!}-\frac{2^2}{3!}+\frac{3^2}{4!}-\cdots$

4) Show that $\sum(-1)^{n-1}\left(\sqrt{n^2+1}-n\right)$ **is conditionally convergent.**

Binomial Theorem for a Rational Index

I. Introduction:

If n is a rational number and $-1 < x < 1$. i.e., $|x| < 1$, the sum of the series

$$1+\frac{nx}{1!}+\frac{n(n-1)}{2!}x^2+\cdots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\cdots$$

is the real positive value of $(1+x)^n$. This series is also represented by $f(n)$. This series is absolutely convergent if $|x| < 1$.

Similarly,

$$f(m)=1+\frac{mx}{1!}+\frac{m(m-1)}{2!}x^2+\cdots\frac{m(m-1)\ldots(m-r+1)}{r!}x^r+\cdots$$

and

$$f(m+n)=1+\frac{(m+n)x}{1!}+\frac{(m+n)(m+n-1)}{2!}x^2+\cdots\frac{(m+n)(m+n-1)\ldots(m+n-r+1)}{r!}x^r+\cdots$$

are absolutely convergent if $|x| < 1$.

$f(m).f(n) = f(m+n)$, for all values of m & n, provided that x is numerically less than unity.

Hence $f(m)$ & $f(n)$ can be multiplied and the resulting series is also an absolutely convergent series.

Moreover, $f(m).f(n).f(p)\ldots s$ factors $= f(m+n+p+\cdots+\text{s terms})$.

Note:

1. The sum to infinity of the binomial series for any given value of x for which is convergent has only one value. Hence $(1+x)^n$ is taken to denote the positive value of $(1+x)^n$. For example, $\left(1+\frac{1}{3}\right)^{-\frac{1}{2}} = \pm\frac{1}{2}\sqrt{3}$ .But only $+\frac{1}{2}\sqrt{3}$ is taken as the value.
2. When $x=1$, the binomial series is convergent if $n>-1$. When $x=-1$, the binomial series is convergent if $n>0$. Hence the binomial expansion is valid for $x=1$, when $n>-1$ and for $x=-1$, when $n>0$.

II. Some important particular cases of binomial expansion:

1. $(1-x)^{-1} = 1+x+x^2+x^3+\cdots$
2. $(1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots+(n+1)x^n+\cdots$
3. $(1-x)^{-3} = \frac{1}{2}\{1.2+2.3x+3.4x^2+4.5x^3\ldots+(n+1)(n+2)x^n+\cdots\}$
4. $(1-x)^{-4} = \frac{1}{6}\{1.2.3+2.3.4.x+3.4.5.x^2+\cdots+n(n+1)(n+2)(n+3).x^n+\cdots\}$
5. $(1-x)^{-n} = 1+\frac{nx}{1!}+\frac{n(n+1)}{2!}x^2+\cdots+\frac{n(n+1)\ldots(n+r+1)}{r!}x^r+\cdots$
6. $(1-x)^{-\frac{1}{2}} = 1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\cdots$
7. $(1-x)^{-\frac{1}{3}} = 1+\frac{1}{3}x+\frac{1.4}{3.6}x^2+\frac{1.4.7}{3.6.9}x^3+\cdots$

Problems:

1. **Find the general term in the expansion of $(1+x)^{\frac{2}{3}}$.**

   Solution:

   The binomial expansion is given by

$$(1+x)^n = 1+\frac{nx}{1!}+\frac{n(n-1)}{2!}x^2+\cdots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\cdots$$

The general term is $(r+1)^{th}$ term = $\frac{n(n-1)...(n-r+1)}{r!}x^r$.

Given: $n = \frac{2}{3}$

$$u_{r+1} = \frac{\left(\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)...\left(\frac{2}{3}-r+1\right)\right)}{r!}x^r$$
$$= \frac{\left(\frac{2}{3}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)...\left(\frac{5-3r}{3}\right)\right)}{r!}x^r$$
$$= \frac{(2.(-1).(-4)...(5-3r))}{3^r r!}x^r$$

$$= \frac{(-1)^{r-1}2.1.4...(3r-5)}{3^r r!}x^r$$

2. **Expand (a)** $(1+3x)^{\frac{5}{2}}$ **given |x|** $< \frac{1}{3}$
   (b) $(a^3 - 2a^2x)^{\frac{5}{3}}$ **in ascending powers of x(TRY IT).**
   Solution:
   (a) The function can be expanded in ascending powers of x if $|3x| < 1$, if $|x| < \frac{1}{3}$

$$(1+3x)^{\frac{5}{2}} = 1 + nC_1(3x) + nC_2(3x)^2 + \cdots + nC_n(3x)^n \quad (\text{since n} = \frac{5}{2})$$
$$(1+3x)^{\frac{5}{2}} = 1 + \frac{\left(\frac{5}{2}\right)(3x)}{1!} + \frac{\left(\frac{5}{2}\right)\left(\frac{5}{2}-1\right)}{2!}(3x)^2 + \cdots + \frac{\frac{5}{2}\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-r+1\right)}{r!}(3x)^r + \cdots$$
$$= 1 + \frac{5}{1!}\left(\frac{3}{2}x\right) + \frac{5.3}{2!}\left(\frac{3}{2}x\right)^2 + \cdots + \frac{5.3.1...(7-2r)}{r!}\left(\frac{3}{2}x\right)^r ...$$

The general term is $(r+1)^{th}$ term
$$= \frac{5.3.1.(-1).(-3).(-5)...(-2r-7)}{r!}\left(\frac{3}{2}x\right)^r$$
$$= (-1)^{r-3}\frac{5.3.1.(1).(3).(5)...(2r-7)}{r!}\left(\frac{3}{2}x\right)^r, r > 3.$$

3. **Find the first term with a negative co-efficient in the expansion of** $(1+2x)^{\frac{14}{3}}$.
   Solution:

The general form is $(r+1)^{th}$ term $= \frac{\left(\left(\frac{14}{3}\right)\left(\frac{14}{3}-1\right)\left(\frac{14}{3}-2\right)...\left(\frac{14}{3}-r+1\right)\right)}{r!}(2x)^r$

The 1^{st} negative term will occur for the least value of r such that

$$\left(\frac{14}{3}-r+1\right) < 0 \Rightarrow \frac{17}{3} - r < 0 \Rightarrow -r < -\frac{17}{3} \Rightarrow r > \frac{17}{3} \Rightarrow r > 5\frac{2}{3}.$$

$i.e., r = 6,$ therefore the value is equal to

$$\frac{\left(\left(\frac{14}{3}\right)\left(\frac{14}{3}-1\right)\left(\frac{14}{3}-2\right)\left(\frac{14}{3}-3\right)\left(\frac{14}{3}-4\right)\left(\frac{14}{3}-5\right)\right)}{6!}(2x)^6$$

$$= \frac{\left(\left(\frac{14}{3}\right)\left(\frac{11}{3}\right)\left(\frac{8}{3}\right)\left(\frac{5}{3}\right)\left(\frac{5}{3}\right)\left(\frac{-1}{3}\right)\right)}{6!}(2x)^6 = \frac{-14.11.8.5.2.1}{6!}\left(\frac{2}{3}\right)^6 x^6$$

4. **Find the greatest term in the expansion of**

   (a) $(1+x)^{\frac{13}{2}}$ **where** $\mathbf{x} = \frac{2}{3}$. **(b)** $(1-x)^{31/3}$**when** $\mathbf{x} = \frac{2}{7}$**.(TRY IT)**

   Solution:

Given $(1+x)^{\frac{13}{2}}$ & $x = \frac{2}{3}, n = \frac{13}{2}$

$$U_{r+1} = \left(\frac{\left(\frac{13}{2}\right)\left(\frac{13}{2}-1\right)\left(\frac{13}{2}-2\right)\ldots\left(\frac{13}{2}-r+1\right)}{r!}\right)x^r \text{ \&}$$

$$U_r = \left(\frac{\left(\frac{13}{2}\right)\left(\frac{13}{2}-1\right)\left(\frac{13}{2}-2\right)\ldots\left(\frac{13}{2}-r+2\right)}{(r-1)!}\right)x^{r-1}$$

$$\Rightarrow \frac{U_{r+1}}{U_r} = \frac{\left(\frac{\left(\frac{13}{2}\right)\left(\frac{13}{2}-1\right)\left(\frac{13}{2}-2\right)\ldots\left(\frac{13}{2}-r+1\right)}{r!}\right)x^r(r-1)!}{\left(\frac{\left(\frac{13}{2}\right)\left(\frac{13}{2}-1\right)\left(\frac{13}{2}-2\right)\ldots\left(\frac{13}{2}-r+2\right)}{(r-1)!}\right)x^{r-1}\,r!}$$

$$\Rightarrow \frac{U_{r+1}}{U_r} = \frac{\left(\frac{15}{2}-r\right)}{r}\left(\frac{2}{3}\right) = \frac{15-2r}{3r} \geq 1 \Rightarrow \frac{U_{r+1}}{U_r} \geq 1 \Rightarrow U_{r+1} \geq U_r$$

Now,

$$\frac{15-2r}{3r} \geq 1 \Rightarrow 15-2r \geq 3r \Rightarrow 15 \geq 5r \Rightarrow r \leq 3.$$

The value of the greatest term $(r=3) =$

$$\frac{\left(\frac{13}{2}\right)\left(\frac{13}{2}-1\right)}{2!}x^2 \Rightarrow \frac{13.11}{2!}\left(\frac{1}{3}\right)^2.$$

III. Approximate values:

1. **Find, correct to six values of decimals, the value of** $\frac{1}{(9998)^{\frac{1}{4}}}$.

Solution:

Given

$$\frac{1}{(9998)^{\frac{1}{4}}} = \frac{1}{(10000-2)^{\frac{1}{4}}} = \frac{1}{(10^4-2)^{\frac{1}{4}}} = \frac{1}{10\left(1-\frac{2}{10^4}\right)^{\frac{1}{4}}} = \frac{1}{10}\left(1-\frac{2}{10^4}\right)^{\frac{1}{4}}$$

$$n = \frac{1}{4}, x = -\frac{2}{10^4}$$

$$\Rightarrow \frac{1}{(9998)^{\frac{1}{4}}} = \frac{1}{10}\left(1 + \left(-\frac{1}{4}\right)\left(-\frac{2}{10^4}\right) + \frac{\left(\left(-\frac{1}{4}\right)\left(-\frac{1}{4}-1\right)\right)}{2!}\left(-\frac{2}{10^4}\right)^2 + \cdots\right)$$

$$\Rightarrow \frac{1}{(9998)^{\frac{1}{4}}} = \frac{1}{10}\left(1 + \left(\frac{1}{2\times 10^4}\right) + \frac{\left(\left(\frac{1}{4}\right)\left(\frac{5}{4}\right)\right)}{2!}\left(\frac{4}{10^8}\right) + \cdots\right)$$

$$\Rightarrow \frac{1}{(9998)^{\frac{1}{4}}} = \frac{1}{10} + \left(\frac{1}{2\times 10^5}\right) + \left(\frac{5}{8\times 10^9}\right)$$

$$\Rightarrow \frac{1}{(9998)^{\frac{1}{4}}} = 0.1 + 0.000005 + 0.000000001 = 0.100005001 \approx 0.100005.$$

2. Calculate, correct to six places of decimals $(1.01)^{\frac{1}{2}} - (0.99)^{\frac{1}{2}}$.

Solution:

Given: $(1.01)^{\frac{1}{2}} - (0.99)^{\frac{1}{2}}$

Consider $(1.01)^{\frac{1}{2}} = (1+0.01)^{\frac{1}{2}} = (1+x)^{\frac{1}{2}}$ and $(0.99)^{\frac{1}{2}} = (1-0.01)^{\frac{1}{2}} = (1-x)^{\frac{1}{2}}$

Now,

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\left(\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\right)}{2!}x^2 + \frac{\left(\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\right)}{3!}x^3 + \cdots \quad ---(1)$$

and

$$(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{\left(\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\right)}{2!}x^2 - \frac{\left(\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\right)}{3!}x^3 + \cdots \quad ---(2)$$

$$(1+x)^{\frac{1}{2}} - (1-x)^{\frac{1}{2}}$$
$$= 1 + \frac{1}{2}x + \frac{\left(\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\right)}{2!}x^2 + \frac{\left(\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\right)}{3!}x^3 + \cdots$$
$$- \left(1 - \frac{1}{2}x + \frac{\left(\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\right)}{2!}x^2 - \frac{\left(\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\right)}{3!}x^3 + \cdots\right)$$

$$\Rightarrow (1)-(2) = x + \frac{x^3}{2\times 2\times 2} + \frac{105x^5}{16\times 120}x^5 = x + \frac{1}{8}x^3 + \frac{105}{1920}x^5$$
$$= x + \frac{1}{8}x^3 + \frac{21}{384}x^5 = x + \frac{1}{8}x^3 + \frac{7}{128}x^5 + \cdots$$

When $x = 0.01$,

$(1)-(2)$

$$= 0.01 + \frac{(0.01)^3}{8} + \frac{7}{128}(0.01)^5$$
$$+ \text{ terms not affecting the eighth decimal places}$$

$$\Rightarrow (1)-(2) = 0.01 + 0.000000125 \approx 0.010000$$

3. **When x is small, prove that $\frac{(1-3x)^{-\frac{2}{3}}+(1-4x)^{-\frac{3}{4}}}{(1-3x)^{-\frac{1}{3}}+(1-4x)^{-\frac{1}{4}}} = 1 + \frac{3}{2}x^2 + 4x^2.(appro)$**

Solution:

$$\frac{(1-3x)^{-\frac{2}{3}}+(1-4x)^{-\frac{3}{4}}}{(1-3x)^{-\frac{1}{3}}+(1-4x)^{-\frac{1}{4}}}$$

$$= \frac{1+\frac{2}{3}(3x)+\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)(-3x)^2+\cdots+1+\frac{3}{4}(4x)+\frac{\left(\left(-\frac{3}{4}\right)\left(-\frac{3}{4}-1\right)\right)}{2!}(-4x)^2+\cdots}{1+\frac{1}{3}(3x)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)(-3x)^2}{2!}+\cdots+1+\frac{1}{4}(4x)+\frac{\left(\left(-\frac{1}{4}\right)\left(-\frac{1}{4}-1\right)\right)}{2!}(-4x)^2+\cdots}$$

$$\Rightarrow \frac{(1-3x)^{-\frac{2}{3}}+(1-4x)^{-\frac{3}{4}}}{(1-3x)^{-\frac{1}{3}}+(1-4x)^{-\frac{1}{4}}} = \frac{1+2x+5x^2+\cdots+1+3x+\frac{21}{2}x^2+\cdots}{1+x+2x^2+\cdots+1+x+\frac{5}{2}x^2+\cdots}$$

$$= \frac{\left(2\left(1+\frac{5}{2}x+\frac{31}{4}x^2\right)\right)}{2\left(1+x+\frac{9}{4}x^2\right)} = \left(1+\frac{5}{2}x+\frac{31}{4}x^2\right)\left(1+x\left(1+\frac{9}{4}x\right)\right)^{-1}$$

$$= 1+\frac{5}{2}x+\frac{31}{4}x^2-x-\frac{5}{2}x^2-\frac{9}{4}x^2+x^2 = 1+\frac{3}{2}x+4x^2$$

4. **Show that $\sqrt{x^2+16}-\sqrt{x^2+9} = \frac{7}{2x}$ nearly for sufficiently large values of x. (TRY IT)**

IV. Exponential and Logarithmic Series

The exponential theorem:

For all values of x, $\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots\frac{1}{n!}\right)^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots$

$$\Rightarrow e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots$$

Note:

$$\boldsymbol{a^x = 1+\frac{x}{1!}log_e a+\frac{x^2}{2!}(log_e a)^2+\cdots+\frac{x^n}{n!}(log_e a)^n+\cdots}$$

For $a^x = e^{log_e a^x} = e^{x\log_e a}$.

1. **Show that the co-efficient of x^n in the infinite series $1+\frac{b+ax}{1!}+\frac{(b+ax)^2}{2!}+\cdots$ is $\frac{e^b a^n}{n!}$.**

Solution:

$1+\frac{b+ax}{1!}+\frac{(b+ax)^2}{2!}+\cdots\frac{(b+ax)^n}{n!} = e^{b+ax} = e^b.e^{ax} = e^b(1+\frac{ax}{1!}+\left(\frac{ax}{2!}\right)^2+\cdots\left(\frac{ax}{n}\right)^n+\cdots)$

The coefficient of x^n is $\frac{e^b a^n}{n!}$

2. **Prove that the co-efficient of x^n in the expansion of $1+\frac{1+2x}{1!}+\frac{(1+2x)^2}{2!}+\cdots$ is $\frac{2^b e}{n!}$.(TRY IT)**

3. **Find the coefficient of x^n in the expansion of $\frac{(1+2x-3x^2)}{e^x}$. (TRY IT)**

4. **Find the sum of the series $1+\frac{1+3}{2!}+\frac{1+3+3^2}{3!}+\frac{1+3+3^2+3^3}{4!}+\cdots\infty$**

Solution:

Given $1+\frac{1+3}{2!}+\frac{1+3+3^2}{3!}+\frac{1+3+3^2+3^3}{4!}+\cdots\infty$

Let $U_n=\frac{1+3+3^2+\cdots+3^{n-1}}{n!}$

$$\Rightarrow U_n=\frac{3^n-1}{3-1}\cdot\frac{1}{n!}=\frac{1}{n!}\frac{3^n-1}{2}=\frac{1}{n!}\left(\frac{3^n}{2!}-\frac{1}{2}\right)$$

Then

$$U_1=\frac{1}{1!}\left(\frac{3}{2}-\frac{1}{2}\right);U_2=\frac{1}{2!}\left(\frac{3^2}{2}-\frac{1}{2}\right);U_3=\frac{1}{3!}\left(\frac{3^3}{2}-\frac{1}{2}\right)$$

and

$$S=\frac{1}{2}\left\{\frac{3}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+\cdots+\frac{3^n}{n!}+\cdots\right\}-\frac{1}{2}\left\{\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\cdots\right\}$$

$$\Rightarrow S=\frac{1}{2}\{e^3-1\}-\frac{1}{2}\{e-1\}=\frac{\{e^3-1-e+1\}}{2}=\frac{\{e^3-e\}}{2}=\frac{e\{e^2-1\}}{2}$$

5. **Find the sum of the series (a) $\sum_{n=0}^{\infty}\frac{(n+1)^3}{n!}x^n$ (b) $\frac{1^2}{1!}+\frac{1^2+2^2}{2!}+\frac{1^2+2^2+3^2}{3!}+\cdots$ *(TRY IT)***

Solution:

Consider $\sum_{n=0}^{\infty}\frac{(n+1)^3}{n!}x^n$.

Put $(n+1)^3=A+Bn+Cn(n-1)+Dn(n-1)(n-2)$

Put $\mathrm{n}=0\Rightarrow\mathrm{A}=1;\ \mathrm{n}=1\Rightarrow\mathrm{B}=7;\mathrm{n}=2\Rightarrow\mathrm{C}=6;\mathrm{n}=3\Rightarrow\mathrm{D}=1.$

Let the sum of the series be (S)

$$\sum_{n=0}^{\infty}\frac{(n+1)^3}{n!}x^n=S=\sum_{n=0}^{\infty}\left(\frac{1+7n+6n(n-1)+n(n-1)(n-2)}{n!}\right)x^n.$$

$$\Rightarrow S=\sum_{n=0}^{\infty}\frac{1}{n!}x^n+\sum_{n=0}^{\infty}\frac{n}{n!}x^n+6\sum_{n=0}^{\infty}\frac{n(n-1)}{n!}x^n+\sum_{n=0}^{\infty}\frac{n(n-1)(n-2)}{n!}x^n.$$

$$\Rightarrow S=\sum_{n=0}^{\infty}\frac{1}{n!}x^n+\sum_{n=0}^{\infty}\frac{1}{(n-1)!}x^n+6\sum_{n=0}^{\infty}\frac{1}{(n-2)!}x^n+\sum_{n=0}^{\infty}\frac{1}{(n-3)!}x^n.$$

Now,

$$\sum_{n=0}^{\infty}\frac{1}{n!}x^n=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\cdots=e^x$$

$$\sum_{n=0}^{\infty}\frac{1}{(n-1)!}x^n=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots=x\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\cdots\right)=xe^x$$

$$\text{Similarly } \sum_{n=0}^{\infty} \frac{1}{(n-2)!} x^n = x^2 e^x; \sum_{n=0}^{\infty} \frac{1}{(n-3)!} x^n = x^3 e^x$$
$$\Rightarrow S = e^x + 7xe^x + 6x^2 e^x + x^3 e^x = e^x(1 + 7x + 6x^2 + x^3).$$

V. Modification of the logarithmic series

If $-1 < x < 1, i.e., |x| < 1$

$$\text{We have } \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots. \qquad ---(1)$$
$$\text{We have } -\log(1+x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots. \qquad ---(2)$$

Adding (1) and (2), we have,

$$\log(1+x) - \log(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$
$$\Rightarrow \frac{\log(1+x)}{\log(1-x)} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

Note:

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Solved Problems:

1. **If $x > 0$, then prove that $\log x = \frac{x-1}{x+1} + \frac{1}{2}\frac{(x^2-1)}{(x+1)^2} + \frac{1}{3}\frac{x^3-1}{(x+1)^3} + \cdots$**

Solution:

$$\frac{x-1}{x+1} + \frac{1}{2}\frac{(x^2-1)}{(x+1)^2} + \frac{1}{3}\frac{x^3-1}{(x+1)^3} + \cdots$$
$$= \left(\frac{x}{x+1} + \frac{1}{2}\frac{x^2}{(x+1)^2} + \frac{1}{3}\frac{x^3}{(x+1)^3} + \cdots\right)$$
$$-\left(\frac{1}{x+1} + \frac{1}{2}\frac{1}{(x+1)^2} + \frac{1}{3}\frac{1}{(x+1)^3} + \cdots\right)$$
$$\Rightarrow \frac{x-1}{x+1} + \frac{1}{2}\frac{(x^2-1)}{(x+1)^2} + \frac{1}{3}\frac{x^3-1}{(x+1)^3} + \cdots = -\log\left(1 - \frac{x}{x+1}\right) + \log\left(1 - \frac{1}{x+1}\right)$$
$$\Rightarrow \frac{x-1}{x+1} + \frac{1}{2}\frac{(x^2-1)}{(x+1)^2} + \frac{1}{3}\frac{x^3-1}{(x+1)^3} + \cdots = -\log\left(\frac{x+1-x}{x+1}\right) + \log\left(\frac{x+1-1}{x+1}\right)$$

$$\Rightarrow \frac{x-1}{x+1} + \frac{1}{2}\frac{(x^2-1)}{(x+1)^2} + \frac{1}{3}\frac{x^3-1}{(x+1)^3} + \cdots = -\log\left(\frac{1}{x+1}\right) + \log\left(\frac{x}{x+1}\right)$$
$$\text{Thus } \frac{x-1}{x+1} + \frac{1}{2}\frac{(x^2-1)}{(x+1)^2} + \frac{1}{3}\frac{x^3-1}{(x+1)^3} + \cdots = \log\left(\frac{\left(\frac{x}{x+1}\right)}{\left(\frac{1}{x+1}\right)}\right) = \log x.$$

And this is valid if

$$\left|\frac{x}{x+1}\right| < 1 \ \& \ \left|\frac{1}{x+1}\right| < 1 \Rightarrow \left|\frac{x}{x+1}\right| < 1 \ \& \left|\frac{1}{x+1}\right| < 1 \Rightarrow |x+1| > 1 \Rightarrow x > 0.$$

2. **Show that $\log\sqrt{12} = 1 + \left(\frac{1}{2} + \frac{1}{3}\right).\frac{1}{4} + \left(\frac{1}{4} + \frac{1}{5}\right)\frac{1}{4^2} + \left(\frac{1}{6} + \frac{1}{7}\right)\frac{1}{4^3} + \cdots$**

Solution:

$$1+\left(\frac{1}{2}+\frac{1}{3}\right).\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\cdots$$

$$=\left(\frac{1}{2}.\frac{1}{4}+\frac{1}{4}.\frac{1}{4^2}+\frac{1}{6}.\frac{1}{4^3}+\cdots\right)+\left(1+\frac{1}{3}.\frac{1}{4}+\frac{1}{5}.\frac{1}{4^2}+\frac{1}{7}.\frac{1}{4^3}+\cdots\right)$$

$$=\left(\frac{1}{2}.\frac{1}{2^2}+\frac{1}{4}.\frac{1}{2^4}+\frac{1}{6}.\frac{1}{2^6}+\cdots\right)+(1+\frac{1}{3}.\frac{1}{2^2}+\frac{1}{5}.\frac{1}{2^4}+\frac{1}{7}.\frac{1}{2^6}+\cdots$$

$$\text{Put }\ \frac{1}{2}=x$$

$$R.H.S=\left(\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\cdots\right)+\left(1+\frac{x^2}{3}+\frac{x^4}{5}+\frac{x^6}{7}+\cdots\right)$$

$$=\left(\frac{x^2}{2}+\frac{(x^2)^2}{4}+\frac{(x^2)^3}{6}+\cdots\right)+\left(1+\frac{x^2}{3}+\frac{(x^2)^2}{5}+\frac{(x^2)^3}{7}+\cdots\right)$$

$$=-\frac{1}{2}\log(1-x^2)+\frac{1}{x}.\frac{1}{2}.\log\frac{1+x}{1-x}$$

Now, Put $x=\frac{1}{2}$

$$\Rightarrow -\frac{1}{2}\log\left(1-\frac{1}{4}\right)+\log\left(\frac{\left(1+\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)}\right)=-\frac{1}{2}\log\left(\frac{3}{4}\right)+\log 3=-\frac{1}{2}\log\left(\frac{3}{4}\right)+\frac{1}{2}\log 9$$

$$=\frac{1}{2}\log 12=\log\sqrt{12}$$

3. **Find the sum of the series $\sum_{n=0}^{\infty}\frac{1}{(2n-1)(2n)(2n+1)}$.**

Solution:

$$\text{let } \mathrm{U_n}=\frac{1}{(2n-1)(2n)(2n+1)}$$

$$\text{Now } \mathrm{U_n}=\frac{1}{(2n-1)(2n)(2n+1)}=\frac{A}{2n-1}+\frac{B}{2n}+\frac{C}{2n+1}\qquad ---(*)$$

$$\Rightarrow 1=A(2n)(2n+1)+B(2n-1)(2n+1)+C(2n-1)(2n)$$

$$\text{Solving the above equation, we have A}=\frac{1}{2};B=-1;C=\frac{1}{2}.$$

$$(*)\Rightarrow U_n=\frac{1}{2}.\frac{1}{2n-1}-\frac{1}{2n}+\frac{1}{2}.\frac{1}{2n+1}$$

$$\text{Now } U_1=\frac{1}{2}.\frac{1}{1}-\frac{1}{2}+\frac{1}{2}.\frac{1}{3};U_2=\frac{1}{2}.\frac{1}{3}-\frac{1}{4}+\frac{1}{2}.\frac{1}{5};U_3=\frac{1}{2}.\frac{1}{5}-\frac{1}{6}+\frac{1}{2}.\frac{1}{7}$$

$$\Rightarrow Sum\ of\ the\ series=\frac{1}{2}.\frac{1}{1}-\frac{1}{2}+\frac{1}{2}.\frac{1}{3}+\frac{1}{2}.\frac{1}{3}-\frac{1}{4}+\frac{1}{2}.\frac{1}{5}+\frac{1}{2}.\frac{1}{5}-\frac{1}{6}+\frac{1}{2}.\frac{1}{7}+\cdots$$

$$=-\frac{1}{2}+\log 2.$$

4. **Show that $\frac{5}{1.2.3}+\frac{7}{3.4.5}+\frac{9}{5.6.7}+\cdots=-1+3log2$ (TRY IT)**

5. **If a, b, c denote three consecutive integers, then show that**

$$log_e b=\frac{1}{2}log_e a+\frac{1}{2}log_e c+\frac{1}{2ac+1}+\frac{1}{3}\frac{1}{(2ac+1)^3}.$$

Solution:

$$\text{RHS} = \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2}\log_e \frac{1+\frac{1}{2ac+1}}{1-\frac{1}{2ac+1}}$$

$$= \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2}\log_e \frac{2ac+2}{2ac}$$

$$= \frac{1}{2}\log ac + \frac{1}{2} + \frac{1}{2}\log \frac{ac+1}{ac} = \frac{1}{2}\log ac \,.\, \frac{ac+1}{ac} = \frac{1}{2}\log(ac+1).$$

If a, b, c denote the three consequtive integers, then $b = a + 1$ & $b = c - 1$.

$a = b - 1; c = b + 1$ & hence $\mathrm{ac} = \mathrm{b}^2 - 1 \Rightarrow ac + 1 = b^2$.

$$\Rightarrow \frac{1}{2}\log(ac+1) = \frac{1}{2}\log b^2 = \log b.$$

Hence proved.

I. Summation by difference series

Consider the series $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$.

From this series, a series of differences, $u_2 - u_1, u_3 - u_2, u_4 - u_3, \dots u_n - u_{n-1}, u_{n+1} - u_n, \dots$ may be constructed. Hence we can write,

$$u_2 - u_1 = \Delta u_1$$
$$u_3 - u_2 = \Delta u_2$$
$$u_4 - u_3 = \Delta u_3$$
$$\dots$$
$$\dots$$
$$u_n - u_{n-1} = \Delta u_{n-1}$$
$$u_{n+1} - u_n = \Delta u_n$$

and call the series $\Delta u_1, \Delta u_2, \dots, \Delta u_{n-1}, \Delta$. This is the series of first differences.

Similarly, writing $\Delta u_n\, for\, \Delta u_{n+1} - \Delta u_n$, we call the series $\Delta^2 u_1, \Delta^2 u_2, \dots, \Delta^2 u_{n-1}, \Delta^2 u_n$. This is the series of second differences.

In a similar way, we can form the series of $3^{rd}, 4^{th}, 5^{th}, \dots, k^{th}$ differences.

II. Successive differences series

Let U_n be a polynomial of degree "k" in "n".

Let the polynomial be $a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n_1 + a_0$ and

$$\Delta U_n = U_{n+1} - U_n$$
$$\Rightarrow \Delta U_n = [a_k (n+1)^k + a_{k-1}(n+1)^{k-1} + \cdots + a_1(n+1) + a_0] - [a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$$
$$\Rightarrow \Delta U_n = a_k[(n+1)^k - n^k] + a_{k-1}[(n+1)^{k-1} - n^{k-1}] + \cdots + a_1$$
$$\Rightarrow \Delta U_n = a_k[kC_1 n^{k-1} + kC_2 n^{k-2} + \cdots] + a_{k-1}[(k-1)C_1 n^{k-2} + \cdots] + a_1$$

Thus, ΔU_n is a polynomial of degree $\mathrm{k}-1$ in "n"

$\Delta^2 U_n$ is a polynomial of degree $\mathrm{k}-2$ in "n"

…

$\Delta^{\mathrm{k}} U_n$ is a polynomial of degree $\mathrm{k}-\mathrm{k}$, a constant.

$\therefore$ $(\mathrm{k}+1)^{\mathrm{th}}$ difference series consist of zero only.

For example, consider the series whose general term is $n^3 + n$.

Hence the series U_n is 2,10,30,68,130,222,350, ...

ΔU_n is 8,20,38,62,92, ...

$\Delta^2 U_n$ is 12,18,24,30,36, ...

$\Delta^3 U_n$ is 6,6,6,6, ...

$\Delta^4 U_n$ is 0,0,0,0, ...

Problems:

1) **Determine the general term and sum up to "n" terms of the series $\mathbf{4 + 14 + 30 + 52 + 80 + 114 + \cdots}$**

Solution:

$$\sum u_n = 4 + 14 + 30 + 52 + 80 + 114 + \cdots$$
$$\sum \Delta u_n \text{ is } 10,16,22,28,34, \dots$$
$$\sum \Delta^2 \mathrm{u_n} \text{ is } 6,6,6,6, \dots$$
$$\sum \Delta^3 \mathrm{u_n} \text{ is } 0,0,0,0 \dots$$

$\therefore u_n$ is a polynomial of degree "2" in "n".

Let u_n be $a_0 + a_1(n-1) + a_2(n-1)(n-2)$.

$$\text{Put } n = 1, u_1 = a_0, a_0 = 4$$
$$n = 2, u_2 = a_0 + a_1 \Rightarrow 14 = 4 + a_1 \Rightarrow a_1 = 10$$
$$n = 2, u_3 = a_0 + 2a_1 + 2a_2 \Rightarrow a_2 = 3$$
$$\text{Thus } u_n = 4 + 10(n-1) + 3(n-1)(n-2)$$
$$S_n = 4\sum 1 + 10\sum (n-1) + 3\sum (n-1)(n-2)$$
$$\Rightarrow S_n = 4n + 10\frac{n(n-1)}{2} + 3\frac{n(n-1)(n-2)}{3}$$
$$\Rightarrow S_n = 4n + 5n(n-1) + n(n-1)(n-2)$$
$$\Rightarrow S_n = 4n + 5n^2 - 5n + n^3 - 3n^2 + 2n$$
$$\Rightarrow S_n = n + n^2(n+2)$$
$$\Rightarrow S_n = n(n+1)^2.$$

2) **Find the n^{th} term and sum up to "n" terms of the series $10, 11, 14, 21, 34, 55, 86, \ldots$**
Solution:

$$\sum u_n = 10,11,14,21,34,55,86, \ldots$$
$$\sum \Delta u_n \text{ is } 1,3,7,13,21,31, \ldots$$
$$\sum \Delta^2 \mathrm{u_n} \text{ is } 2,4,6,8,10, \ldots$$
$$\sum \Delta^3 \mathrm{u_n} \text{ is } 2,2,2,2 \ldots$$
$$\sum \Delta^4 \mathrm{u_n} \text{ is } 0,0,0,0 \ldots$$
$\therefore u_n$ is a polynomial of degree "3" in "0".
Let u_n be $a_0 + a_1(n-1) + a_2(n-1)(n-2)$.
$$\text{Put } n = 1, u_1 = a_0, a_0 = 10$$
$$n = 2, u_2 = a_0 + a_1 \Rightarrow 11 = 10 + a_1 \Rightarrow a_1 = 1$$
$$n = 3, u_3 = a_0 + 2a_1 + 2a_2 \Rightarrow a_2 = 1 \;\&\; a_3 = \frac{1}{3}$$
$$\text{Thus } u_n = 10 + (n-1) + (n-1)(n-2) + \tfrac{1}{3}(n-1)(n-2)(n-3)$$
$$S_n = \sum 10 + 10\sum (n-1) + \sum (n-1)(n-2) + \frac{1}{3}\sum (n-1)(n-2)(n-3)$$
$$\Rightarrow S_n = 10n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{3} + \frac{1}{3}\frac{(n-1)(n-2)(n-3)}{4}$$
$$\Rightarrow S_n = 10n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{3} + \frac{1}{12}(n-1)(n-2)(n-3).$$

3) **Determine the n^{th} term and sum up to $k^4 n^4$ terms of the series $6 + 9 + 14 + 23 + 40 + \cdots$**
Solution:
$$\sum u_n = 6 + 9 + 14 + 23 + 40 + \cdots$$
$$\sum \Delta u_n \text{ is } 3,5,9,17, \ldots$$
$$\sum \mathrm{u_n} \text{ is a G.P of common ratio 2.}$$
$$\sum \mathrm{u_n} \text{ is of the form a}.2^{n-1} + x, \text{a polynomial of degree one.}$$
$$\text{Let } \mathrm{u_n} = a.2^{n-1} + a_0 + a_1(n-1)$$

$$\text{Put } n = 1, u_1 = a + a_0, a + a_0 = 6 \quad ---(1)$$
$$n = 2, u_2 = a_0 + a_1 + 2a \Rightarrow 9 = a_0 + a_1 + 2a \quad ---(2)$$
$$n = 3, u_3 = a_0 + a_2^2 + a_1 \Rightarrow 4a + a_0 + 2a_1 = 14 \quad ---(3)$$
$$\text{On solving } (1), (2) \ \& \ (3), a = 2, a_1 = 1$$
$$\Rightarrow \mathrm{u_n} = 2.2^{n-1} + 4 + n - 1 \Rightarrow u_n = 2^n + n + 3$$
$$S_n = \sum 2^n + \sum n + \sum 3$$
$$\Rightarrow S_n = 2 + 2^2 + 2^3 + \cdots + 2^n + \frac{n(n-1)}{3} + 3n$$
$$\Rightarrow S_n = 2^{n+1} - 2 + \frac{n(n-1)}{3} + 3n.$$

Do it:

1) **Find the n^{th} term and sum up to "n" terms of the following series:**
   (a) $\mathbf{1, 2, 3, 6, 17, 54, 171, \ldots}$
   (b) $\mathbf{2 + 5 + 12 + 31 + 86 + 249 + \cdots}$
   (c) $\mathbf{10 + 23 + 60 + 169 + 494 + \cdots}$
   (d) $\mathbf{9 + 22 + 59 + 168 + 493 + \cdots}$
   (e) $\mathbf{7 + 10 + 14 + 20 + 30 + 48 + 82 + \cdots}$
   (f) $\mathbf{4 + 10 + 20 + 35 + 56 + 84 + 120 + \cdots}$

III. Recurrence series

Problems:

1) **Find the n^{th} term of the recurring series $3 + 4x + 6x^2 + 10x^3 + \cdots$**

Solution:

The given series has four terms. So let the scale of relation be $1 + px + qx^2$.

$$\text{The generating function is } \frac{a + bx}{1 + px + qx^2}$$
$$\Rightarrow \frac{a + bx}{1 + px + qx^2} = 3 + 4x + 6x^2 + 10x^3 + \cdots$$
$$\Rightarrow a + bx = (1 + px + qx^2)(3 + 4x + 6x^2 + 10x^3 + \cdots)$$
$$\Rightarrow a + bx = 3 + 4x + 6x^2 + 10x^3 + 3px + 4px^2 + 6px^3$$
$$+10px^4 + 3qx^2 + 4qx^3 + 6qx^4 + 10qx^5$$
$$\Rightarrow a + bx = 3 + (4 + 3p)x + (6 + 4p + 3q)x^2 + (10 + 6p + 4q)x^3 + \cdots$$
$$a = 3; b = 4 + 3p; 6 + 4p + 3q = 0; 10 + 6p + 4q = 0 \quad ---(1)$$
$$\text{On solving the above equation, we have}, p = -3; q = 2; b = -5; A = -1; B = -2.$$
$$\Rightarrow \frac{3 - 5x}{(2x - 1)(-1)} = -\frac{1}{1 - 2x} + \frac{2}{1 - x}$$
$$\Rightarrow \frac{3 - 5x}{(2x - 1)(-1)} = (1 - 2x)^{-1} + 2(1 - x)^{-1}$$
$$\Rightarrow \frac{3 - 5x}{(2x - 1)(-1)} = 1 + 2x + (2x)^2 + \cdots + (2x)^n + \cdots + 2(1 + x + x^2 + \cdots + x^n + \cdots)$$

The general term of U_n = coefficient of x^n in the expansion $=2^n + 2$.

2) **Find the n^{th} term of the recurring series**
   (a) $\mathbf{1 + 3 + 7 + 13 + 21 + 31 + \cdots}$ [Hint: Consider the given as $1 + 3x + 7x^2 + 13x^3 + 21x^4 + 31x^5 + \cdots$]
   (b) $\mathbf{2 + 6 + 14 + 30 + \cdots}$
   (c) $\mathbf{2 + 7x + 25x^2 + 91x^3 + \cdots}$
   (d) $\mathbf{1 + 5x + 9x^2 + 13x^3 + \cdots}$
   (e) $\mathbf{2 + 3x + 5x^2 + 9x^3 + \cdots}$